Aspects of the integer quantum Hall effect

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Abstract

We review some of the concepts which shaped the understanding of the integer quantum Hall effect, as well as the mathematical results they led to. Also described are the underlying physical motivations. Emphasis is placed on the equivalence of different approaches.

To Barry, with admiration

1 Introduction

The integer quantum Hall effect (IQHE) has been at the crossroads of several developments in mathematical physics, such as non-commutative geometry, index theory, localization, and the adiabatic theorem. Some of the most illuminating contributions to these topics are coauthored by Barry Simon. The fractional quantum Hall effect, which we shall not review here, further ties to conformal and topological field theory [21] and to the classification of certain integral lattices [19].

The IQHE appears in some two-dimensional samples at temperatures close to zero and in a strong, transverse magnetic field. The current density j and the stationary electric field E, both lying in the plane of the sample, are empirically related by the Hall-Ohm law

$$j = \sigma E \; , \tag{1}$$

which introduces the conductivity tensor σ. The phenomenon is that, under appropriate conditions, transport is dissipationless, $j \cdot E = 0$, which means that the tensor is antisymmetric,

$$\sigma = \begin{pmatrix} 0 & \sigma_{\mathrm{H}} \\ -\sigma_{\mathrm{H}} & 0 \end{pmatrix} \; ; \tag{2}$$

even more remarkably, the Hall conductance σ_{H} is quantized in multiples of e^2/h, where e is the electron charge and h is Planck's constant (i.e., of $1/2\pi$ if natural units are used); and the quantization of σ_{H} is accurate to within 10^{-8} when the magnetic field or the density of electrons are varied over a sizeable range, a fact called a plateau. Its width may be comparable to the separation between plateaus.

Three basically distinct explanations have been proposed for the IQHE. Common to all, at least in their original formulation, is that they treat the electrons as independent particles, except for the Pauli principle.

(i) The Hall conductance may be identified with the charged pumped across a closed ribbon when the magnetic flux threading it is increased by one flux quantum [26]. The Hamiltonians before and after that change are unitarily conjugate and the state has evolved adiabatically in between because the transport is dissipationless. The occupation of states follows the spectral flow as a function of the flux and that charge is the number of occupied eigenvalues having crossed the Fermi energy in the process. Clearly, that number is an integer.

(ii) Linear response theory computes the current j induced by a weak electric field E in the bulk of the sample, which yields the Kubo formula for the Hall conductance. That expression can be related to a Chern number [33], which is an integer.

(iii) The Hall current is ascribed to states flowing at the edge of the sample. In special cases it may be identified with the number of edge channels [23, 10], which is an integer.

The Hall conductance as defined on the basis of (i) has been linked [26, 23] to the current flowing *along* the ribbon when an electric potential is applied *across* it. That current may be interpreted as flowing either in the bulk [26] or at the edges [23] of the ribbon. In real experiments it is a combination of both possibilities [23].

Following their original formulation, the approaches (i-iii) have not only gained in mathematical precision, but the physical concepts involved have been clarified as well. For instance, the argument [26] for (i) depended on the fact that eigenvalues moving down (resp. up) under the spectral flow are associated with the inner (resp. outer) edge. If the Hamiltonian is symmetric against rotations along the ribbon, as it was first assumed, then the eigenvalues are monotone in the flux throughout its variation, permitting the conclusion on charge transport stated above. If it is not, as e.g. implied by the presence of weak disorder, eigenvalue crossings formerly protected by symmetry may now be avoided, destroying monotonicity. This issue, which reflects the possibility of eigenstates tunnelling between edges, admittedly remained to be investigated in [26]. It can be avoided altogether if the outer edge is pushed to infinity [4], which turns the ribbon into a punctured plane. In that geometry the Hall conductance can be identified with the relative index [4] of a pair of projections. As another example, the argument [33] for (ii) assumes a periodic Hamiltonian and hence rational magnetic flux per unit cell and no disorder. This is however not needed, since the unit cell may be replaced by a torus of boundary conditions [29] or fluxes [5] (in the latter case the approach extends to interacting particles). Another generalization [6] of the Chern character, more appropriate to the thermodynamic limit, is by means of non-commutative geometry.

Disorder is crucial for the formation of plateaus. There, the Fermi energy varies within in an interval where bulk states are localized, a so-called mobility gap. Such a variation changes the occupation of states (and hence the electron density), but not that of those participating in transport. By contrast, a variation over the extended states spectrum, or over a spectral gap, changes both, resp. neither. Disorder thus affects the analysis in two main ways: First, it destroys a symmetry, like the above mentioned rotations or periodicity. This applies even if the simplifying assumption is made that (a) the Fermi energy falls in a spectral gap. Second (b), the Fermi energy actually falls in a mobility gap. It it therefore of utmost importance that definitions of Hall conductance and their equivalence be compatible with disorder. For (i, ii) this was achieved in [6, 4]; for (iii, a) in [31, 24] and for (iii, b) in [16].

2 Results

We shall momentarily present three definitions of Hall conductance related to the above pictures (i-iii). A mathematical setting, in which they are conveniently placed, is that of discrete Schrödinger operators [14]. The bulk is represented by the lattice $\mathbb{Z}^2 \ni x = (x_1, x_2)$ with Hamiltonian $H_B = H_B^*$ on $\ell^2(\mathbb{Z}^2)$. Its matrix elements $H_B(x, x')$, $(x, x' \in \mathbb{Z}^2)$, are of short-range in the sense that

$$\sup_{x\in\mathbb{Z}^2} \sum_{x'\in\mathbb{Z}^2} |H_B(x,x')|(\mathrm{e}^{\mu|x-x'|} - 1) =: C_1 < \infty \tag{3}$$

for some $\mu > 0$, where $|x| = |x_1| + |x_2|$. A bounded, open interval $\Delta \subset \mathbb{R}$, which shall contain the Fermi energy, is assumed to lie (a) in a *spectral gap* or, more generally, (b) in a *mobility gap*:

(a)
$$\Delta \cap \sigma(H_B) = \varnothing \; ; \tag{4}$$

(b) For some $\nu > 0$,

$$\sup_{g\in B_1(\Delta)} \sum_{x,x'\in\mathbb{Z}^2} |g(H_B)(x,x')|(1+|x|)^{-\nu}\mathrm{e}^{\mu|x-x'|} < \infty \; , \tag{5}$$

where $B_1(\Delta)$ denotes the set of Borel measurable functions g which are constant in $\{\lambda | \lambda < \Delta\}$ and in $\{\lambda | \lambda > \Delta\}$ with $|g(\lambda)| \le 1$ for all $\lambda \in \mathbb{R}$. In particular, the spectrum is pure-point in Δ [25]. Denoting by E_M the characteristic function of $M \subset \mathbb{R}$, the assumption is completed by

$$\dim E_{\{\lambda\}}(H_B) < \infty \; , \qquad (\lambda \in \Delta) \; , \tag{6}$$

i.e., no eigenvalue in Δ is infinitely degenerate.

Condition (5) is basically a statement about dynamical localization. It has been established in [1] and more explicitely in [30], where the above property is related to the SULE property, as well as in [2], where g is allowed to be constant, rather than zero, outside of Δ. The condition holds true almost surely for ergodic Schrödinger operators whose Green's function $G(x, x'; z) = (H_B - z)^{-1}(x, x')$ satisfies a moment condition [3] of the form

$$\limsup_{E\in\Delta,\eta\to 0} \mathbb{E}(|G(x,x';E+\mathrm{i}\eta)|^s) \le C\mathrm{e}^{-\mu|x-x'|}$$

for some $0 < s < 1$, but can also be proved [22] using multi-scale analysis [20]. Condition (6) appears to be essential for a plateau, in view of the fact that for the Landau Hamiltonian (though defined on the continuum rather than on the lattice) σ_{H} jumps as the Fermi energy crosses an infinitely degenerate Landau level. The condition, in fact simple spectrum (almost surely), follows from the arguments in [32]. We stress however that the hypotheses (a) and (b) themselves are deterministic. Translation covariance or ergodicity of H_B are not assumed here.

The three definitions of Hall conductance shall now be associated with a Flux, the Bulk and the Edge. The physical motivations relating them to the approaches (i-iii) mentioned before will be given in the next section.

(i) The definition is based on the index of a pair of projections and depends on some *unitaries* associated with gauge transformations. Let P and Q be two orthogonal projections on a Hilbert space, so that $P - Q$ is compact. Then

$$\mathrm{Ind}(P, Q) = \dim\{\psi \mid P\psi = \psi, Q\psi = 0\} - \dim\{\psi \mid P\psi = 0, Q\psi = \psi\} . \tag{7}$$

Let $U(x)$, $(x \in \mathbb{Z}^2)$, satisfy $|U(x)| = 1$ and

$$|U(x) - U(y)| \le C_1 \frac{|x - y|}{1 + |x|} , \qquad (|x - y| \le C_2 |x|), \tag{8}$$

for some C_1, C_2. Along a large loop encircling the origin counterclockwise, the phases of $U(y)/U(x)$, which are small for single bonds (x, y), add up to a multiple, $N(U) \in \mathbb{Z}$, of 2π. We assume that the winding number $N(U)$ equals 1. Then

$$\sigma_F(\lambda) = \frac{1}{2\pi} \mathrm{Ind}(P_\lambda, U P_\lambda U^*) , \tag{9}$$

where $P_\lambda = E_{(-\infty,\lambda)}(H_B)$.

(ii) The definition makes use of *switch functions*: Let $\Lambda(n)$, $(n \in \mathbb{Z})$, be a function which equals 0 for large negative n, resp. 1 for large positive n. Then

$$\sigma_B(\lambda) = \mathrm{i} \, \mathrm{tr} \, P_\lambda \big[[P_\lambda, \Lambda_1], [P_\lambda, \Lambda_2] \big] , \tag{10}$$

where Λ_1 and Λ_2 are switch functions of x_1, resp. x_2.

(iii) The sample with an edge is modeled as a half-plane $\mathbb{Z} \times \mathbb{Z}_a$, where $\mathbb{Z}_a = \{n \in \mathbb{Z} \mid n \le a\}$, with the height a of the edge eventually tending to ∞. The Hamiltonian $H_a = H_a^*$ on $\ell^2(\mathbb{Z} \times \mathbb{Z}_a)$ is obtained by restriction of H_B under some largely arbitrary local *boundary condition*, described as follows. Denoting by $\mathcal{J}_a : \ell^2(\mathbb{Z} \times \mathbb{Z}_a) \to \ell^2(\mathbb{Z}^2)$ the extension by 0, we assume that

$$E_a = \mathcal{J}_a H_a - H_B \mathcal{J}_a : \ell^2(\mathbb{Z} \times \mathbb{Z}_a) \to \ell^2(\mathbb{Z}^2)$$

satisfies

$$\sup_{\substack{x \in \mathbb{Z}^2 \\ a \in \mathbb{Z}}} \sum_{x' \in \mathbb{Z} \times \mathbb{Z}_a} |E_a(x, x')| \mathrm{e}^{\mu(|x_2 + a| + |x_1 - x_1'|)} < \infty . \tag{11}$$

We then set

$$\sigma_E = \mathrm{i} \lim_{\eta\to 0} \lim_{a\to\infty} \operatorname{tr} \rho'(H_a)[H_a, \Lambda_1] A_{\eta,a}(\Lambda_2) \ , \tag{12}$$

where $\rho \in C^\infty(\mathbb{R})$ satisfies

$$\rho(\lambda) = \begin{cases} 1\ , & (\lambda < \Delta)\ , \\ 0\ , & (\lambda > \Delta)\ , \end{cases} \tag{13}$$

and

$$A_{\eta,a}(X) = \eta \int_0^\infty \mathrm{e}^{-\eta t} \mathrm{e}^{\mathrm{i}H_a t} X \mathrm{e}^{-\mathrm{i}H_a t} dt$$

is the average over a time $\sim \eta^{-1}$ of a bounded operator X with respect to the Heisenberg evolution generated by H_a.

The main results are as follows: First of all, these quantities are well-defined for $\lambda \in \Delta$. Second, they are independent of various auxiliary objects, such as U, Λ_1, Λ_2 and E_a. In particular, they do not change when U, Λ_1, or Λ_2 are replaced by some translates. For concreteness only, the reader may think of $U_p(x) = U(x-p)$ with

$$U(x) = \frac{x}{|x|} \tag{14}$$

and $p = (p_1, p_2) \in \mathbb{Z}^{2*} = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$, as well as $\Lambda_{i,p} = \theta(x_i - p_i)$; for Dirichlet boundary conditions, $H_a = \mathcal{J}_a^* H_B \mathcal{J}_a$, the locality condition (11) holds true by (3). Third, the conductances are independent of $\lambda \in \Delta$, resp. ρ with (13), which is the manifestation of a plateau. Fourth,

$$\sigma_F = \sigma_B = \sigma_E \ .$$

Finally, $2\pi\sigma_F$ is manifestly an integer.

Some of these definitions were first formulated under slightly different assumptions. For instance, under assumption (a) the definition (12) of the edge Hall conductance can be replaced by the simpler

$$\sigma_E = \mathrm{i} \operatorname{tr} \rho'(H_a)[H_a, \Lambda_1] \ , \tag{15}$$

for any $a \in \mathbb{Z}$. In an ergodic setting, Λ_1 and Λ_2 in (10) may be replaced by x_1, resp. x_2 and the trace by the trace per unit area,

$$\sigma_B = \mathrm{i} \operatorname{tr}_\infty P_\lambda \big[[P_\lambda, x_1], [P_\lambda, x_2] \big] \ , \tag{16}$$

which takes an almost sure value [7]. The same replacement can be made in (15), with the trace now becoming trace per unit length [31].

It is now appropriate to review the genesis of these results. The definition (16) was proposed in [6] and identified as a 2-cocycle. It was shown to be an integer by relating it to a Fredholm index, through a formula of Connes [13]. In [4] that index was formulated as the index (7) of a pair of projections and endowed with independent physical motivation. It was proved to be equal to the expectation of (10), again in the ergodic setting and using Connes' formula, but without explicit reference to non-commutative geometry.

Translation invariance of (9) was shown and used in [6, 4]. It was noted in [16] that the same is true for (10), and not just for its expectation, which allowed to turn $\sigma_F = \sigma_B$ into a deterministic statement. In [31], which is placed in the ergodic setting and under assumption (a), σ_E was displayed as a winding number, which provided an independent reason for its quantization. It was also shown to be a 1-cocycle, from which the conclusion $\sigma_E = \sigma_B$ was drawn using K-theory. The equality $\sigma_E = \sigma_F$, again for (a) but in a deterministic setting, was shown in [15, 27]. In the more general case (b), $\sigma_E = \sigma_B$ is due to [16], with a special case due to [12]. In all this, the pioneering role of non-commutative geometry [6, 31] is manifest.

3 Motivations

We provide the physical pictures underlying the definitions (i-iii) of Hall conductances. The discussion is largely heuristic.

(i) In one of its guises the Laughlin argument relates the Hall conductance with the charge pulled from infinity when a flux quantum is slowly added near the origin. By the continuity equation, the charge Q inside a loop $\mathcal{C}$ changes at the rate

$$\frac{dQ}{dt} = -\oint_{\mathcal{C}} j \cdot \nu ds ,$$

where ν is the outward normal. By Faraday's law, a change in the flux Φ is accompanied by an electric field,

$$\oint_{\mathcal{C}} E \cdot \tau ds = -\frac{d\Phi}{dt} ,$$

where τ is the tangent vector. If the change is slow, the field is nearly stationary, so that (1, 2) apply, resulting in $\Delta Q = \sigma_{\mathrm{H}} \Delta\Phi$. This is to be compared with the quantum mechanical computation of ΔQ. The state of the system is $U(t,0)P_\lambda U(t,0)^*$, where $U(t,0)$ is the evolution generated by the Hamiltonian, now depending on time through the flux. By the time, $t = t_0$, the flux has increased by $\Delta\Phi = 2\pi$ the Hamiltonian becomes unitarily conjugate to H through $U(x)$ with winding number 1. Its Fermi projection is $UP_\lambda U^*$. In the limit of a large loop and of a slow process the change ΔQ equals the excess number of electrons in the evolved many-body state, as compared with the ground state for the same flux:

$$\Delta Q = \mathrm{Ind}(U(t_0,0)P_\lambda U(t_0,0)^*, UP_\lambda U^*) = \mathrm{Ind}(P_\lambda, UP_\lambda U^*) .$$

The second equation follows because of the additivity ($\mathrm{Ind}(P,R) = \mathrm{Ind}(P,Q) + \mathrm{Ind}(Q,R)$) and the continuity ($\|P-Q\| < 1 \Rightarrow \mathrm{Ind}(P,Q) = 0$) of the index, implying

$$\mathrm{Ind}(U(t,0)P_\lambda U(t,0)^*, P_\lambda) = 0 .$$

(ii) The Kubo formula for conductance is derived by adiabatically switching an electric field and considering the linear response of the system. The function $-\Lambda_2$ can be seen as an electric potential of unit drop for a field pointing in the positive x_2-direction, while the operator $\mathrm{i}[H, \Lambda_1]$ stands for the total current in the x_1-direction. The expectation of the latter in the state perturbed by the former yields σ_{12}, i.e., σ_{H} in eq. (2). The time-dependent Hamiltonian is

$$H(t) = H - \Lambda_2 f(t)\ , \qquad (t \le 0)\ ,$$

where $f(t)$ slowly interpolates between 0 and 1, e.g., $f(t) = \mathrm{e}^{\eta t}$ for some small $\eta > 0$. As in (i) the unperturbed density matrix is P_λ. The perturbed density matrix $\rho(t)$ satisfies the initial value problem

$$\frac{d}{dt}\rho(t) = -\mathrm{i}[H(t), \rho(t)]\ , \qquad \lim_{t\to-\infty} \mathrm{e}^{\mathrm{i}Ht}\rho(t)\mathrm{e}^{-\mathrm{i}Ht} = P_\lambda\ .$$

To first order in the electric field the solution is

$$\rho(0) - P_\lambda = \mathrm{i}\int_{-\infty}^{0} dt\, \mathrm{e}^{\eta t}\mathrm{e}^{\mathrm{i}Ht}[\Lambda_2, P_\lambda]\mathrm{e}^{-\mathrm{i}Ht}\ ,$$

and we obtain, after an integration by parts,

$$\sigma_B = \lim_{\eta\to 0} \mathrm{i}\,\mathrm{tr}[H, \Lambda_1](\rho(0) - P_\lambda) = \lim_{\eta\to 0} \mathrm{i}\eta\,\mathrm{tr}\int_{-\infty}^{0} dt\, \mathrm{e}^{\eta t}(\mathrm{e}^{-\mathrm{i}Ht}\Lambda_1\mathrm{e}^{\mathrm{i}Ht} - \Lambda_1)[\Lambda_2, P_\lambda]\ . \tag{17}$$

Since P_λ is a projection we have $[\Lambda_2, P_\lambda] = P_\lambda[\Lambda_2, P_\lambda](1 - P_\lambda) + (1 - P_\lambda)[\Lambda_2, P_\lambda]P_\lambda$. The substitution of this into (17) amounts, by cyclicity, to the substitution of Λ_1 there by the expression

$$(1 - P_\lambda)\Lambda_1 P_\lambda + P_\lambda\Lambda_1(1 - P_\lambda) = \big[[\Lambda_1, P_\lambda], P_\lambda\big]\ .$$

The l.h.s. contributes two terms to (17) containing $\mathrm{e}^{\pm\mathrm{i}Ht}$, similar to one another, one of which is

$$\mathrm{i}\eta\,\mathrm{tr}\int_{-\infty}^{0} dt\, \mathrm{e}^{\eta t}\mathrm{e}^{-\mathrm{i}Ht}(1 - P_\lambda)\Lambda_1 P_\lambda \mathrm{e}^{\mathrm{i}Ht}[\Lambda_2, P_\lambda]\ .$$

As we now sketch, it vanishes for $\eta \to 0$. Representing the propagators as $\mathrm{e}^{-\mathrm{i}Ht} = \int \mathrm{e}^{-\mathrm{i}\mu t} dP_\mu$, we are led to

$$\mathrm{i}\eta\int_{-\infty}^{0} dt\, \mathrm{e}^{\eta t}\mathrm{e}^{-\mathrm{i}(\mu_+ - \mu_-)t} = -\frac{\eta}{\mu_+ - \mu_- + \mathrm{i}\eta} \tag{18}$$

with $\mu_+ \ge \lambda$, $\mu_- < \lambda$. Since this quantity vanishes pointwise as $\eta \to 0$ for (μ_+, μ_-) in the stated region, one is tempted to conclude that

$$\sigma_B = -\mathrm{i}\,\mathrm{tr}\big[[\Lambda_1, P_\lambda], P_\lambda\big][\Lambda_2, P_\lambda] = \mathrm{i}\,\mathrm{tr}\, P_\lambda\big[[\Lambda_1, P_\lambda], [\Lambda_2, P_\lambda]\big]\ ,$$

which is (10). The passage from (17) to (10), or rather its analogue in the ergodic setting, cf. (16), has been put on a firm basis, see [7], but also [2, 9]; in the present setting the result can be obtained using methods of [17]. In both cases it is crucial that λ lies in a mobility gap, which allows to control the small denominator in (18).

(iii) For a simpler start let us first discuss the definition (15) for σ_E valid in the case of a spectral gap. We interpret $\rho(H_a)$ as the 1-particle density matrix of a stationary quantum state. Though some current is flowing near the edge we should discard it, as it is supposed to be canceled by current flowing at an opposite edge located at $x_2 = -\infty$. The idea is that the two opposite edges of a macroscopic sample are infinitely separated from a microscopic perspective, and we focus on one of them. The drop in electrical potential used in (ii) is now given a different physical realization: The Fermi energy is lowered by δ at the first edge, but not at the second. Then a net current $I = \mathrm{i}\,\mathrm{tr}((\rho(H_a+\delta) - \rho(H_a))[H_a, \Lambda_1])$ is flowing, resulting in

$$\sigma_E = \lim_{\delta\to 0} \frac{I}{\delta} = \mathrm{i}\,\mathrm{tr}\,\rho'(H_a)[H_a, \Lambda_1]\ .$$

The current operator $\mathrm{i}[H_a, \Lambda_1]$ is relevant only on states along a strip near $x_1 = 0$, and $\rho'(H_a)$ only near the edge $x_2 = a$, because $\rho'(H_B) = 0$ due to (4). The intersection of the two strips is compact, which is basically why the trace exists. In presence of a mobility gap, however, this property of $\rho'(H_a)$ fails. In search of a proper definition of σ_E for this case, we consider only the current flowing across the line $x_1 = 0$ within a finite window $0 \le x_2 < a$ next to the edge. This amounts to modifying the current operator to be $\mathrm{i}[H_a, \Lambda_1]\Lambda_2$ (or a symmetrized version thereof), with which one may tentatively use

$$\lim_{a\to\infty} \mathrm{i}\,\mathrm{tr}\,\rho'(H_a)[H_a, \Lambda_1]\Lambda_2 \tag{19}$$

as a definition for σ_E. Though this limit exists, it is not the physically correct choice. States in the range of $\rho'(H_a)$ supported far away from the edge are close to bound states of the bulk Hamiltonian, $H_B\psi_\lambda = \lambda\psi_\lambda$, or linear combinations thereof. Such states may carry persistent currents (whence the operator in (15) is no longer trace class), but no current across the line $x_1 = 0$, since $(\psi_\lambda, [H_B, \Lambda_1]\psi_\lambda) = 0$. This cancellation is the rationale for ignoring the part $x_2 < 0$ of the line $x_1 = 0$ by means of the cutoff Λ_2 in (19), however the cancellation is not achieved on states located near the end point $x = (0, 0)$. The contribution missed by (19) is $(\psi_\lambda, \mathrm{i}[H_B, \Lambda_1](1-\Lambda_2)\psi_\lambda) = -(\psi_\lambda, \mathrm{i}[H_B, \Lambda_1]\Lambda_2\psi_\lambda)$ from each bound state. By weighting them with $\rho'(\lambda)$ we amend the definition (19) of the edge conductance:

$$\sigma_E = \lim_{a\to\infty} \mathrm{i}\,\mathrm{tr}\,\rho'(H_a)[H_a, \Lambda_1]\Lambda_2 - \mathrm{i}\sum_{\lambda\in\mathcal{E}_\Delta} \rho'(\lambda)\,\mathrm{tr}\,E_{\{\lambda\}}[H_B, \Lambda_1]\Lambda_2 E_{\{\lambda\}}\ . \tag{20}$$

The sum, which is over the eigenvalues in Δ, happens to be absolutely convergent, but there is no general reason for it to vanish. In fact, it can be shown to be non-zero for the Harper Hamiltonian with a Cauchy distributed random potential.

Alternatively, one may use in (19) and instead of Λ_2 a cutoff which commutes with the dynamics generated by H_a, at least in some limit. Its use will not create spurious contributions which call for compensation. Such a possibility is realized by the time average of Λ_2 and leads to definition (12). Unlike (20), it is stated purely in terms of the edge Hamiltonian H_a. Nevertheless the two definitions agree [16]. We also remark that the time averages in eqs. (12) and (17), though of different physical origin, are mathematically related, which is instrumental to the proof of $\sigma_E = \sigma_B$.

4 Some snapshots from the proofs

1. The equality $\sigma_F = \sigma_B$ in case (b)

In the case of finite dimensional projections P and Q the index can be computed as $\mathrm{Ind}(P,Q) = \mathrm{tr}(P-Q)$. The generalization [4] to the infinite dimensional case is

$$\mathrm{Ind}(P,Q) = \mathrm{tr}(P-Q)^{2n+1}$$

if $P - Q \in \mathcal{J}_{2n+1}$ for some odd integer $2n+1$, where $\mathcal{J}_p$, $(1 \le p < \infty)$, are the trace ideals. In particular, $\mathrm{tr}(P-Q)^3 = \mathrm{tr}(P-Q)$ if $P - Q \in \mathcal{J}_1$, which can be seen from the identity

$$(P-Q) - (P-Q)^3 = [PQ, QP] = [PQ, [Q, P-Q]] \ . \tag{21}$$

In the application to the QHE, where $P = P_\lambda$, $Q = UP_\lambda U^*$, the difference luckily is not trace class, since the contrary would imply $\mathrm{tr}(P_\lambda - UP_\lambda U^*) = 0$ by evaluating the trace in the position basis. However, the third power of this difference is trace class, which yields

$$\sigma_F = \frac{1}{2\pi}\,\mathrm{tr}(P_\lambda - UP_\lambda U^*)^3 = \frac{\mathrm{i}}{\pi}\sum_{x,y,z\in\mathbb{Z}^2} P_\lambda(x,y)P_\lambda(y,z)P_\lambda(z,x)S(x,y,z)\ , \tag{22}$$

$$S(x,y,z) = -\frac{\mathrm{i}}{2}\Big(1 - \frac{U(x)}{U(y)}\Big)\Big(1 - \frac{U(y)}{U(z)}\Big)\Big(1 - \frac{U(z)}{U(x)}\Big)\ .$$

To show its equality with

$$\sigma_B = \mathrm{i}\sum_{x,y,z\in\mathbb{Z}^2} P_\lambda(x,y)P_\lambda^\perp(y,z)P_\lambda(z,x)[(\Lambda_1(y)-\Lambda_1(x))(\Lambda_2(z)-\Lambda_2(y)) - (1\leftrightarrow 2)]\ , \tag{23}$$

the authors of [4] assumed that the projection is ergodic (or covariant) w.r.t. magnetic translations. Translation invariance of σ_F then implies that (22) is a translation invariant function of the randomness and a.s. equal to its expectation. After taking the expectation of both equations (22, 23), the expressions $\mathbb{E}(P_\lambda(x,y)P_\lambda^{(\perp)}(y,z)P_\lambda(z,x))$ are constant under a common shift a of the summation variables x, y, z. By trading one them against a, the sums over the latter involve only U, resp. Λ_i; moreover, for $U = x/|x|$, they are related through a formula of Connes, to be discussed below. The result is $\sigma_F = \mathbb{E}(\sigma_B)$. A slightly different use of translation invariance, which does not depend on ergodicity, was made in [16]. Let $U = U_p$ and $\Lambda_i = \theta_{i,p}$ as in (14). Since in fact both σ_F and σ_B are independent of p, averaging of (22, 23) over $\Lambda_L^* = \{p \in \mathbb{Z}^{2*} \mid |p| \le L\}$ results in

$$\sigma_F = \frac{\mathrm{i}}{\pi L^2}\sum_{\substack{p\in\Lambda_L^*\\ x,y,z\in\mathbb{Z}^2}} P_\lambda(x,y)P_\lambda(y,z)P_\lambda(z,x)S(p,x,y,z)\ , \tag{24}$$

$$S(p,x,y,z) = \sin\angle(x,p,y) + \sin\angle(y,p,z) + \sin\angle(z,p,x)\ , \tag{25}$$

respectively in

$$\sigma_B = \frac{\mathrm{i}}{L^2}\sum_{\substack{p\in\Lambda_L^*\\ x,y,z\in\mathbb{Z}^2}} P_\lambda(x,y)P_\lambda^\perp(y,z)P_\lambda(z,x)\cdot$$

$$\cdot\,[(\theta(y_1-p_1)-\theta(x_1-p_1))(\theta(z_2-p_2)-\theta(y_2-p_2)) - (1\leftrightarrow 2)]\ . \tag{26}$$

These two expressions, which do not depend on L by derivation, will be shown to be equal in the limit $L \to \infty$. Because the decay (5) applies to P_λ, the summation ranges $x \in \mathbb{Z}^2$, $p \in \Lambda_L^*$ can then be replaced by $x \in \Lambda_L$, $p \in \mathbb{Z}^{2*}$. At this point Connes' formula [13]

$$\frac{1}{\pi} \sum_{p \in \mathbb{Z}^{2*}} S(p, x, y, z) = 2 \, \mathrm{Area}(x, y, z) \tag{27}$$

may be used in (24), where $\mathrm{Area}(x, y, z)$ is the triangle's oriented area, namely $\frac{1}{2}(x-y) \wedge (y-z)$. On the other hand the corresponding sum in (26) also yields $2 \, \mathrm{Area}(x, y, z)$, since

$$\sum_{p_i \in \mathbb{Z}^*} (\theta(y_i - p_i) - \theta(x_i - p_i)) = y_i - x_i \ .$$

The proof of $\sigma_F = \sigma_B$ is completed by $P_\lambda^\perp(y, z) = \delta_{yz} - P_\lambda(y, z)$.

2. Connes' formula

The role of the sine function in (27, 25) is less special than one might think, as noted by [11]: For a fixed triplet $u^{(1)}$, $u^{(2)}$, $u^{(3)} \in \mathbb{Z}^2$, let $\alpha_i(p) = \angle(u^{(i+1)}, p, u^{(i+2)}) \in (-\pi, \pi)$ be the angle of view from $p \in \mathbb{Z}^{2*}$ of $u^{(i+2)}$ relative to $u^{(i+1)}$ (with $\alpha_i(p) = 0$ if p lies between them). Let $g(\alpha)$ be a bounded function satisfying $g(-\alpha) = g(\alpha)$ and

$$g(\alpha) = \alpha + \mathrm{O}(\alpha^3) \tag{28}$$

near $\alpha = 0$. Then,

$$\sum_{p \in \mathbb{Z}^{2*}} \sum_{i=1}^{3} g(\alpha_i(p)) = 2\pi \, \mathrm{Area}(u^{(1)}, u^{(2)}, u^{(3)}) \ . \tag{29}$$

The proof is as follows. We may assume the triangle to be positively oriented. The statement (29) is true for $g(\alpha) = \alpha$. Indeed, for each $p \in \mathbb{Z}^{2*}$,

$$\sum_{i=1}^{3} \alpha_i(p) = 2\pi \left\{ \begin{array}{c} 1 \\ 1/2 \\ 0 \end{array} \right\} \text{for } p \left\{ \begin{array}{c} \text{inside} \\ \text{on the boundary of} \\ \text{outside} \end{array} \right\} \text{the triangle.} \tag{30}$$

Thus, for $g(\alpha) = \alpha$ the l.h.s. of (29) is $2\pi\times$ the number of dual lattice sites within the triangle (counting a boundary site with weight 1/2). This number equals the triangle's area.

The above observation reduces (29) to the statement that for $f(\alpha) = g(\alpha) - \alpha$

$$\sum_{p \in \mathbb{Z}^2} \sum_{i=1}^{3} f(\alpha_i(p)) = 0 \ . \tag{31}$$

A significant difference between f and g is that the individual terms $f(\alpha_i(p))$ are summable in $p \in \mathbb{Z}^2$, since by (28) $f(\alpha_i(p)) = \mathrm{O}(|p|^{-3})$ for $|p| \to \infty$. However, each of the three individual sums changes sign under the reflection with respect to the midpoint of the corresponding edge, $(u^{(i+1)} + u^{(i+2)})/2 \in (\mathbb{Z}/2)^2$ (which is a symmetry of the lattice $\mathbb{Z}^2$). Thus even the individual sums (at given i) vanish.

3. The equality $\sigma_F = \sigma_E$ in case (a)

In this setting σ_E is given by (15), without the need to pass to the limit $a \to \infty$. We may thus take $H_{a=0}$ on $\ell^2(\mathbb{Z} \times \mathbb{Z}_0)$ as the edge Hamiltonian. Unlike for H_B, the interval Δ is not a spectral gap for H_0. Its spectrum in Δ may actually be absolutely continuous [28, 8, 18], which is a manifestation of states extending along the edge. As a result, the matrix elements of a spectral projection $E_{(-\infty,\lambda)}(H_0)$, $(\lambda \in \Delta)$, will no longer decay rapidly away from the diagonal, which is the property that in the bulk case ensured

$$P - UPU^* \in \mathcal{J}_3 \tag{32}$$

for $P = P_\lambda = E_{(-\infty,\lambda)}(H_B)$ and U as in (8). By contrast, we also have $P_\lambda = \rho(H_B)$ with ρ as in (13) due to assumption (a), and the property (32) extends to $P = \rho(H_0)$. The price to pay is that P is no longer a projection, but that does not seem to be a crucial aspect of the Laughlin argument. Morally, we may identify σ_F with $(2\pi)^{-1}\operatorname{tr}(P - UPU^*)$, while eq. (21) is cautioning us that the correct computation of the trace is, in the case of projections, by discarding the trace of the commutator on the r.h.s., which, though not defined, is formally zero. That identity reads

$$\begin{aligned}
(P-Q)-(P-Q)^3 \;=\;& \frac{1}{2}[PQ, QP] - \frac{1}{2}\big[(1-P)(1-Q), (1-Q)(1-P)\big] \\
&+(1-2P)(P-P^2) - (1-2Q)(Q-Q^2) \\
&+\frac{3}{2}\{P-Q, P-P^2+Q-Q^2\}
\end{aligned}$$

if it is not restricted to projections, where $\{\cdot,\cdot\}$ denotes the anticommutator. It suggests to consider the expression

$$K(U) := \operatorname{tr}\Big(\frac{3}{2}\{P-Q, (P-P^2)+(Q-Q^2)\} + (P-Q)^3\Big) \tag{33}$$

as a replacement for $\operatorname{tr}(P-Q)$ when P and $Q = UPU^*$ are unitarily conjugated. More precisely, we consider unitaries U which are multiplication operators w.r.t. some fixed basis, like the position basis of $\ell^2(\mathbb{Z} \times \mathbb{Z}_0)$, and operators $P = P^*$ such that $(P-Q)^3$, $(P-Q)(P-P^2)$, $p(P) - p(Q) \in \mathcal{J}_1$ for $p(\lambda) = \lambda - \lambda^2$ and $p(\lambda) = (1-2\lambda)(\lambda-\lambda^2)$. We remark that these properties are satisfied for $P = \rho(H_0)$ and for U as in (8), if $U = 1$ on all but a finite piece of the edge. For instance, $(P-Q)(P-P^2)$ is then associated with such a piece, because $P - P^2$ vanishes away from the edge. Hence it is trace class.

The important property is that $K(U)$ is unaffected by changes of U which are trace class. This is used as follows: Let $U - 1$ be supported in a cone whose rays point into the lower half-plane $\mathbb{Z} \times \mathbb{Z}_0$, and let its curl, i.e., the magnetic flux, be concentrated near the vertex. Moving the vertex without changing the fan of the cone is an example of such a change of U. If the vertex too is placed well inside the lower half-plane, the cone does not intersect the edge and the first term in (33) is negligible. In this limit $K(U)$ reduces to $2\pi\sigma_F$. If, on the other hand, the vertex is pulled across the edge and well into the upper half-plane, then the second term in (33) is associated with the intersection of the cone with the lower half-plane, which in the limit is a negligible (though infinite) tail. Moreover, and still inside the lower half-plane, U may be represented as an exponential, which near the edge is of the form $e^{2\pi i \Lambda_1(x)}$. The remaining first term in (33) can then be computed as

$$\begin{aligned}
3\,\mathrm{tr}(P-Q)\big((P-P^2)+(Q-Q^2)\big) &\approx 6\,\mathrm{tr}(P-Q)(P-P^2) \\
&\approx 6\int_0^{2\pi} d\varphi \frac{d}{d\varphi}\,\mathrm{tr}(P-\mathrm{e}^{\mathrm{i}\varphi\Lambda_1}P\mathrm{e}^{-\mathrm{i}\varphi\Lambda_1})(P-P^2) \\
&= 6\cdot 2\pi\mathrm{i}\,\mathrm{tr}[\rho(H_0),\Lambda_1]\big(\rho(H_0)-\rho(H_0)^2\big) \\
&= 6\cdot 2\pi\mathrm{i}\,\mathrm{tr}[H_0,\Lambda_1]\rho'(H_0)\big(\rho(H_0)-\rho(H_0)^2\big) \\
&= 2\pi\mathrm{i}\,\mathrm{tr}[H_0,\Lambda_1]\tilde{\rho}'(H_0) = 2\pi\sigma_E\ ,
\end{aligned}$$

where $\tilde{\rho} = 3\rho^2 - 2\rho^3$ satisfies the same assumption (13) as ρ does.

4. The equality $\sigma_E = \sigma_B$ in case (b)

We sketch some of the steps towards this identity when σ_E is defined by (20). If the other definition (12) is chosen, the argument runs along similar lines. The statement may be rephrased as

$$\lim_{a\to\infty} \mathrm{i}\,\mathrm{tr}\,\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 = \sigma_B + \mathrm{i}\sum_{\lambda\in\mathcal{E}_\Delta}\rho'(\lambda)\,\mathrm{tr}\,E_{\{\lambda\}}[H_B,\Lambda_1]\Lambda_2 E_{\{\lambda\}}\ . \tag{34}$$

The operator on the l.h.s. is geometrically associated with the finite but growing portion $0\le x_2\le a$ of the the line $x_1=0$. It therefore has no chance to converge in trace class norm as $a\to\infty$. To see that its trace nevertheless does, we look for an operator $Z(a)\in\mathcal{J}_1$ with $\mathrm{tr}\,Z(a)=0$, and replace the operator with

$$\mathrm{i}\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 - Z(a)\ , \tag{35}$$

hoping that convergence in that norm now holds true. A first attempt is $Z(a) = \mathrm{i}[\rho(H_a),\Lambda_1]\Lambda_2$, which satisfies the two requirements; in particular, its trace is seen to vanish by computing it in the position basis. A partial cancellation between the two terms is made manifest using the Helffer-Sjöstrand representations

$$\begin{aligned}
\rho(H_a) &= \frac{1}{2\pi}\int d^2z\,\partial_{\bar z}\rho(z)R_a(z)\ ,\\
\rho'(H_a) &= -\frac{1}{2\pi}\int d^2z\,\partial_{\bar z}\rho(z)R_a(z)^2\ ,
\end{aligned}$$

where $R_a(z)=(H_a-z)^{-1}$ and $\rho(z)$ on the r.h.s. is a quasi-analytic extension of $\rho(x)$. It yields in fact

$$\begin{aligned}
[\rho(H_a),\Lambda_1]\Lambda_2 &= -\frac{1}{2\pi}\int d^2z\,\partial_{\bar z}\rho(z)R_a(z)[H_a,\Lambda_1]R_a(z)\Lambda_2\ , \qquad (36)\\
\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 &= -\frac{1}{2\pi}\int d^2z\,\partial_{\bar z}\rho(z)R_a(z)^2[H_a,\Lambda_1]\Lambda_2\ .
\end{aligned}$$

The two expressions would look even more similar if in the second line one power of the resolvent could be moved to the end of the expression. This however is just a commutator which may be absorbed into a redefinition $Z(a)$. Then (35) reads

$$
\begin{aligned}
-\frac{\mathrm{i}}{2\pi}\int d^2z\partial_{\bar z}\rho(z)R_a(z)[H_a,\Lambda_1](\Lambda_2R_a(z)-R_a(z)\Lambda_2)& \\
=-\frac{\mathrm{i}}{2\pi}\int d^2z\partial_{\bar z}\rho(z)R_a(z)[H_a,\Lambda_1]R_a(z)[H_a,\Lambda_2]R_a(z)\ .& \quad (37)
\end{aligned}
$$

This expression is geometrically associated with the intersection of the lines $x_1=0$ and $x_2=0$, which is independent of a. It is therefore reasonable that it has a limit as $a\to\infty$, which is indeed obtained by replacing the subscript a with B. It remains to show that the trace of the bulk quantity T so obtained equals the r.h.s. of (34). To this end we use (37, 36) in reverse, but now with $a\rightsquigarrow B$, and obtain

$$
T=-\mathrm{i}[\rho(H_B),\Lambda_1]\Lambda_2-\frac{\mathrm{i}}{2\pi}\int d^2z\partial_{\bar z}\rho(z)R(z)[H_B,\Lambda_1]\Lambda_2R(z)\ .
$$

Unlike for $a<\infty$, the two terms are not separately trace class. We next compare the expression with (17): While the first term, $\mathrm{e}^{-\mathrm{i}Ht}\Lambda_1\mathrm{e}^{\mathrm{i}Ht}$, is necessary to ensure that the whole expression is trace class, it is formally only the second one which contributes to the trace, as explained there: $\sigma_B=-\mathrm{i}\,\mathrm{tr}\,\Lambda_1[\Lambda_2,P_\lambda]$, or $\sigma_B=\mathrm{i}\,\mathrm{tr}[\Lambda_1,P_\lambda]\Lambda_2$. These expressions are not well defined, but the following is a correct representation for σ_B:

$$
\sigma_B(\lambda_0)=-\mathrm{i}\,\mathrm{tr}\,E_-[P_{\lambda_0},\Lambda_1]\Lambda_2E_- -\mathrm{i}\,\mathrm{tr}\,E_+[P_{\lambda_0},\Lambda_1]\Lambda_2E_+ -\sum_{\lambda\in\mathcal{E}_\Delta}\mathrm{i}\,\mathrm{tr}\,E_{\{\lambda\}}[P_{\lambda_0},\Lambda_1]\Lambda_2E_{\{\lambda\}}\ , \tag{38}
$$

where E_-, E_+ are the spectral projections for H_B onto $\{\lambda\mid\lambda<\Delta\}$, resp. $\{\lambda\mid\lambda>\Delta\}$. Since $\sigma_B(\lambda_0)$ is independent of $\lambda_0\in\Delta$ we may replace P_{λ_0} by $\rho(H_B)$ in (38). We then frame T similarly with $E_\pm$, $E_{\{\lambda\}}$, without changing its trace. The first term is then just σ_B; the contributions with $E_\pm$ from the second vanish because $E_\pm R(z)$ and $R(z)E_\pm$ are analytic on the support of $\rho(z)$ or of $\rho(z)-1$. The remaining contribution is

$$
-\frac{\mathrm{i}}{2\pi}\sum_{\lambda\in\mathcal{E}_\Delta}\int d^2z\partial_{\bar z}\rho(z)(\lambda-z)^{-2}\,\mathrm{tr}\,E_{\{\lambda\}}[H_B,\Lambda_1]\Lambda_2E_{\{\lambda\}}\ ,
$$

which equals the last term in (34).

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