Problem 1. Let $a > 0$ be fixed. Define the function

$$\chi_a(x) = \exp(x \log(a))$$

1. Compute $\chi'_a$. Give a differential equation similar to the one for the exponential function that characterizes $\chi_a$.

2. Compute the limit

$$\lim_{x \to 0} \frac{\chi_a(x) - 1}{x}.$$ 

Remark: It is customary to denote $\chi_a(x) = (\exp(\log(a)))^x = a^x$.

Problem 2. Let $y \in \mathbb{R}$ be fixed. Consider the two functions

$$g_y(x) = \sin(x + y), \quad h_y(x) = \sin(x) \cos(y) + \cos(x) \sin(y)$$

Prove that both functions satisfy the equation

$$f''(x) = -f(x), \quad (x \in \mathbb{R})$$

and that $g_y(0) = h_y(0)$ as well as $g'_y(0) = h'_y(0)$.

Note: Just as with the exponential function, this proves that $g_y(x) = h_y(x)$ for all $x \in \mathbb{R}$, which is one of the standard ‘trigonometric identities’.