Solution 5

Problem 1. (i) We parametrize the path $\alpha = \alpha_1 + \alpha_2$ as
$$\alpha_1(t) = (1 + i)t, \ t \in [0, 1], \quad \alpha_2(t) = (1 - i)t + (1 + i), \ t \in [0, 1],$$
to get
$$\int_{\alpha} z \, dz = \int_0^1 (1 + i)t(1 + i) \, dt + \int_0^1 ((1 + i)t + (1 + i))(1 - i) \, dt = 2,$$
$$\int_{\alpha} \bar{z} \, dz = \int_0^1 (1 - i)t(1 + i) \, dt + \int_0^1 ((1 + i)t + (1 - i))(1 - i) \, dt = 2 - 2i,$$
$$\int_{\alpha} \text{Re}(z) \, dz = \int_0^1 t(1 + i) \, dt + \int_0^1 (t + 1)(1 - i) \, dt = 2 - i.$$

Note that $z = (z^2/2)'$ immediately implies the first result: $\int_{\alpha} z \, dz = 2^2/2 - 0 = 2.$

Problem 2. (i) Along the line segment, elementary geometry yields that $|z| \geq \sqrt{2}/2$ and hence $|z|^2 \leq 2.$
Since the segment itself has length $\sqrt{2},$ we conclude that $|\int_{\alpha} |z|^{-2} \, dz| \leq 2\sqrt{2}.$
(ii) Letting $z = x + iy,$ we have that $|e^{ikz}| = e^{-ky} \leq 1$ since $k > 0, y \geq 0$ on $\alpha_R$ and $|1 + z^2| \geq |1 - |z|^2| = R^2 - 1$ on $\alpha_R.$
Since the length of the path is $\pi R,$ we conclude that $\left| \int_{\alpha_R} e^{ikz}(1 + z^2)^{-2} \, dz \right| \leq (R^2 - 1)^{-1}(\pi R) \rightarrow 0$
as $R \rightarrow +\infty.$
(iii) Along $\gamma_R,$ we have that $|e^{-z^2}| = e^{y^2-x^2} \leq e^{h^2-R^2}$ so that $\left| \int_{\gamma_R} e^{-z^2} \, dz \right| \leq he^{h^2-R^2} \rightarrow 0$ as $R \rightarrow +\infty$ for any fixed $h > 0.$

Problem 3. (i) We first check that in a neighbourhood of $\alpha,$ $\text{Log}(z) = (z\text{Log}(z) - z)'.$ Hence
$$\int_{\alpha} \text{Log}(z) \, dz = (z\text{Log}(z) - z)|_{z=1} = \pi/2 - i + 1.$$

(ii) We parametrize the curves as $\alpha_{\pm}(t) = e^{\pm it}$ where $t \in [0, \pi]$ in both cases. Hence,
$$\int_{\alpha_{\pm}} \bar{z} \, dz = \int_0^\pi e^{\mp it}(\pm i)e^{\pm it} \, dt = \pm i\pi.$$

Note that this difference implies that $z \mapsto \bar{z}$ does not have an antiderivative.

Problem 4. Since
$$\int e^{at}(\cos(bt) + i \sin(bt)) \, dt = \int e^{at} \, dt = \alpha^{-1}e^{at} = \frac{(a - ib)}{a^2 + b^2}e^{at}(\cos(bt) + i \sin(bt)),$$
the first identities follow by taking the real and imaginary parts above.
Finally, we let $\alpha = -a + ib$ for $a > 0$ and $b \in \mathbb{R}$ arbitrary and note that $\lim_{t \rightarrow +\infty} |\cos(bt)e^{-at}| \leq \lim_{t \rightarrow +\infty} e^{-at} = 0$ and similarly $\lim_{t \rightarrow +\infty} |\sin(bt)e^{-at}| = 0,$ while $\cos(bt)e^{-at}|_{t=0} = 1$ and $\sin(bt)e^{-at}|_{t=0} = 0$
to conclude that
$$\int_0^\infty e^{at} \cos(bt) \, dt = -\frac{e^{-at}}{a^2 + b^2} (-a \cos(bt) + b \sin(bt))|_{t=0} = \frac{a}{a^2 + b^2},$$
$$\int_0^\infty e^{at} \sin(bt) \, dt = -\frac{e^{-at}}{a^2 + b^2} (-a \sin(bt) - b \cos(bt))|_{t=0} = \frac{b}{a^2 + b^2}.$$