MATH 305, 2017W

Solution 10

Problem 1. (i) Of the three simple poles at \( z_j = e^{\pi j / 3} e^{2\pi i j / 3}, j = 0, 1, 2 \), only \( z_0 \) lies in the interior of the closed curve, and \( \text{Res}(f; z_0) = \frac{1}{3\pi i} \mid_{z = e^{\pi / 3}} = \frac{1}{3} e^{-\frac{2\pi i}{3}} \). Hence, denoting the arc by \( \alpha_R \), the residue theorem gives

\[ \int_0^R \frac{1}{x^3 + 1} \, dx + \int_{\alpha_R} \frac{1}{z^3 + 1} \, dz - \int_0^R \frac{1}{t^3 + 1} e^{2\pi i} \, dt = \frac{2\pi i}{3} e^{-\frac{2\pi i}{3}}. \]

Now \( \left| \int_{\alpha_R} \frac{1}{z^3 + 1} \, dz \right| \leq \frac{2\pi R}{3} \frac{1}{R^2 - 1} \to 0 \) as \( R \to \infty \), and hence

\[ \int_0^R \frac{1}{x^3 + 1} \, dx = \frac{2\pi i e^{-\frac{2\pi i}{3}}}{3(1 - e^{\frac{2\pi i}{3}})} = \frac{\pi}{3 \sin(\pi/3)} = \frac{2\pi \sqrt{3}}{9}, \]

where we multiplied both the numerator and the denominator by \( e^{-\pi i / 3} \).

(ii) For a general power \( n \), one chooses a sector with opening angle \( 2\pi / n \), which again contains only one pole with residue \( \frac{1}{n} e^{-\pi i (n-1) / n} \). There again, the integrals along the two segments add up since \( (e^{-\pi i / n})^n = 1 \), the second one carrying a factor \( -e^{2\pi i / n} \). Hence, \( \int_0^\infty \frac{1}{x^3 + 1} \, dx = \frac{\pi}{n \sin(\pi / n)}, \quad n \in \mathbb{N}, n \geq 2 \).

Problem 2. Following the method presented in class, we choose the branch of both the logarithm and \( x^a \) defined by the argument ranging in \([0, 2\pi)\). Using the notation from the class,

\[ \left| \int_{\alpha_4} f(z) \, dz \right| \leq 2\pi R^{1-a} \frac{\ln R}{1 - r} \to 0 \quad (r \to 0), \quad \left| \int_{\alpha_2} f(z) \, dz \right| \leq 2\pi R^{1-a} \frac{\ln R}{R - 1} \to 0 \quad (R \to \infty), \]

since \( 1 > 1 - a > 0 \). Here again, for \( x > 0 \),

\[ \lim_{x \to 0^+} (x + ie)^{-a} \log(x + ie) = \frac{\ln x}{x^a}, \quad \lim_{x \to 0^+} (x - ie)^{-a} \log(x - ie) = \frac{\ln x + 2\pi i}{x^a e^{2\pi ia}} \]

since \( \lim_{x \to 0^+} \arg(0, 2\pi)(x \pm ie) = \begin{cases} 0 & \text{(+)} \\ 2\pi & \text{(-)} \end{cases} \). Hence,

\[ \lim_{x \to 0^+} \int_{\alpha_1} f(z) \, dz = \int_0^R \frac{\ln x}{x^a (1 + x)} \, dx, \quad \lim_{x \to 0^+} \int_{\alpha_3} f(z) \, dz = -e^{-2\pi ia} \left( \int_r^R \frac{\ln x}{x^a (1 + x)} \, dx + 2\pi i \int_r^R \frac{1}{x^a (1 + x)} \, dx \right) \]

Finally, since \( \text{Res}(f; -1) = \frac{i\pi}{\sin(\pi)} \), we conclude using the hint

\[ I(1 - e^{-2\pi ia}) = -2\pi^2 e^{-ia} + 2\pi i e^{-2\pi ia} \frac{\pi}{\sin(\pi a)} \]

namely \( i \sin(\pi a) I = -\pi^2 - 2\pi^2 e^{-ia} \frac{\pi}{\sin(\pi a)} \), yielding the claim.

Problem 3. (i) With the Taylor series \( \sin(w) = \sum_{k=0}^\infty \frac{(-1)^k w^{2k+1}}{(2k+1)!} \) and the substitution \( w = \frac{1}{2z} \), we obtain

\[ f(z) = z^3 \sum_{k=0}^\infty \frac{(-1)^k}{(2k + 1)!} (2z)^{-2k-1} = \sum_{k=0}^\infty \frac{(-1)^k}{2^{2k+1}(2k+1)!} z^{-2k} = \frac{z^2}{2 - \frac{1}{48} + \frac{1}{2^5} \frac{1}{z^2} - \cdots} \]

(ii) \( f \) being holomorphic in \( C(2; 0, 1) \), it is analytic and hence has only a regular part: \( f_-(z) = 0 \). From the Laurent series computed in (i), we conclude that in \( C(0; 0, \infty) \)

\[ f_-(z) = \sum_{k=2}^\infty \frac{(-1)^k}{2^{2k+1}(2k+1)!} z^{-2k-2} = \frac{1}{2^5} \frac{1}{z^2} - \frac{1}{2^7} \frac{1}{z^4} + \cdots \]

(iii) From (i), we read off \( \text{Res}(f; 0) = 0 \) and hence \( f_{|z|=1} f(z) \, dz = 0 \).