Problem 1. Compute the residues at the singularities of the following functions:

(i) \( f_1(z) = z^2 e^{1/z} \),  
(ii) \( f_2(z) = \frac{\cos(2z)}{z(z - \pi/2)^2} \).

*Hint:* For \( f_1 \), use the Taylor series of the exponential function.

(iii) Compute \( \oint_{|z|=1} f_i(z)\,dz \), for both \( i = 1, 2 \).

Problem 2. (i) Compute

\[
\int_{|z|=2\pi} \tan(z)\,dz
\]

*Hint:* You can use directly that \( \text{Res} \left( \frac{f(z)}{g(z)}; z_0 \right) = \frac{f(z_0)}{g'(z_0)} \) when \( z_0 \) is a simple zero of \( g \).

(ii) Compute

\[
I = \int_{-\infty}^{+\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx
\]

*Hint:* Use a rectangular closed curve with corners \( R, R + i, -R + i, -R \).

Problem 3. Let \( P \) be a polynomial of degree \( d \), and let \( \alpha \) be a closed, positively oriented simple curve such that no zero of \( P \) lies on \( \alpha \). Show that

\[
\frac{1}{2\pi i} \oint_{\alpha} \frac{P'(z)}{P(z)}\,dz = N,
\]

where \( 0 \leq N \leq d \) is an integer counting the number of zeroes of \( P \) inside \( \alpha \) (with multiplicity: a zero of multiplicity \( m \) is counted \( m \) times)

*Hint:* One can write \( P(z) = c(z - z_1) \cdots (z - z_d) \); why?

Problem 4. Show that \( \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \) by following the strategy presented in class but using \( f(z) = \frac{1}{z^2} \) directly (no limit to be taken at the end of the argument). Adapt the computation to account for the fact that the integrand now has a pole of order three at \( z = 0 \).

*Hint:* You can use that \( g(z) = (\pi z) \cot(\pi z) \) is such that \( \lim_{z\to0} g''(z) = -\frac{2\pi^2}{3} \).

Problem 5. Compute

\[
I = \int_{0}^{2\pi} \frac{\sin^2(\varphi)}{5 + 4\cos(\varphi)}\,d\varphi.
\]