Problem 1. Let $f$ be holomorphic in the open annulus $\{ z \in \mathbb{C} : 1 < |z| < 2 \}$ and continuous on its closure $\{ z \in \mathbb{C} : 1 \leq |z| \leq 2 \}$. Assume that $|f(z)| \leq 12$ for all $z$ such that $|z| = 2$ and that $|f(z)| \leq 3$ for all $z$ such that $|z| = 1$. By considering the function $g(z) = f(z)/3z^2$, prove that $|f(z)| \leq 3|z|^2$.

Problem 2. Assume that the $3 \times 3$ matrix $A$ has a characteristic polynomial

$$c_A(t) = -t^3 + t^2 + 4t - 24.$$ 

Prove that $A$ does not have any eigenvalue in the disk $\{ z \in \mathbb{C} : |z| \leq 2 \}$.

Problem 3. Let $w \in \mathbb{C}$ be such that $|w| < 1$.

(i) Show that if $z \in \mathbb{C}$ is such that $|z| = 1$ then

$$\left| \frac{z - w}{1 - \overline{w}z} \right| = 1.$$

(ii) Use (i) to show that

$$\left| \frac{z - w}{1 - \overline{w}z} \right| < 1$$

for all $z \in \mathbb{C}$ is such that $|z| < 1$. *Hint:* Why is the condition $|w| < 1$ needed?