**Problem 1.** Give the matrix of the following linear transformation with respect to the canonical bases of \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \). \( \varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \), with \( \varphi \left( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \begin{pmatrix} 2x - 3z \\ -x - y - z \end{pmatrix} \).

**Problem 2.** Let
\[
\varphi : M_{2 \times 2} \rightarrow \mathbb{R} \\
A \mapsto \text{Tr}(A)
\]
be the trace map.
(i) Prove that \( \varphi \) is linear.
(ii) Compute its matrix with respect to the canonical basis of \( M_{2 \times 2} \) and \{1\}.
(iii) For any two matrices \( A, B \in M_{2 \times 2} \), let
\[
[A, B] = AB - BA
\]
be their **commutator**. Prove that \( [A, B] \in \mathcal{N}(\varphi) \) for any \( A, B \in M_{2 \times 2} \).

**Problem 3.** Let
\[
\varphi : M_{2 \times 2} \rightarrow M_{2 \times 2} \\
A \mapsto A^T
\]
be the transposition map (see Assignment 4, Problem 3).
(i) Compute its matrix in the canonical basis of \( M_{2 \times 2} \).
(ii) Compute its matrix in the following basis:
\[
E = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.
\]

**Problem 4.** A linear map \( \varphi : V \rightarrow W \) between two vector spaces of the same dimension is called **invertible** if there exists a linear map \( \psi : W \rightarrow V \) such that \( \varphi \circ \psi = \text{id}_W \) and \( \psi \circ \varphi = \text{id}_V \), where \( \text{id}_V \) is the identity map, \( \text{id}_V(x) = x \) for all \( x \in V \), \( \text{id}_W(y) = y \) for all \( y \in W \).

Let \( E \), respectively \( F \), be a basis of \( V \), respectively \( W \). We denote the inverse by \( \varphi^{-1} \).

Prove that the matrix \( (\varphi)^E_F \) is invertible if the map \( \varphi \) is invertible, with matrix inverse given by
\[
[(\varphi)^E_F]^{-1} = (\varphi^{-1})^F_E.
\]

**Problem 5.** Let \( Q : V \rightarrow V \) be a projection, namely \( Q \circ Q = Q \) (see Assignment 4, Problem 1).

Exhibit a basis \( E = \{e_1, \ldots, e_n\} \) and a \( k \in \{1, \ldots, n\} \) such that
\[
(Q)^E_E = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}
\]
where \( I_k \in M_{k \times k} \) is the identity matrix, and the other blocks are \( k \times (n-k), (n-k) \times k, (n-k) \times (n-k) \) matrices of zeroes.