Problem 1. (i) Let $V_1, V_2, V_3$ be three real vector spaces and let $\varphi : V_1 \to V_2$ and $\psi : V_2 \to V_3$ be two linear transformations. Prove that their composition

$$\psi \circ \varphi : V_1 \to V_3$$

$$x \mapsto (\psi \circ \varphi)(x) = \psi(\varphi(x))$$

is a linear transformation.

(ii) Assume that $V = W_1 \oplus W_2$ and let

$$P : V \to V$$

$$x = x_1 + x_2 \mapsto P(x) = x_1$$

Prove that $P \circ P = P$. Characterize the kernel $N(P)$ and the range $R(P)$.

(iii) Let $Q : V \to V$ be a linear transformation such that $\mathcal{N}(Q) = \{0\}$. Prove that $V = \mathcal{N}(Q) \oplus R(Q)$ (Hint: $x = x - Q(x) + Q(x)$)

Problem 2.

Let $V$ be a finite-dimensional vector space.

(i) Let $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ be two bases of $V$. Prove that there is a unique linear transformation $\varphi : V \to V$ such that $\varphi(e_i) = f_i$.

(ii) Let $\psi : V \to V$ be a linear transformation such that $\mathcal{N}(\psi) = \{0\}$. Prove that $\{\psi(e_1), \ldots, \psi(e_n)\}$ is another basis of $V$.

Problem 3. For any matrix $M \in \mathcal{M}_{p \times n}$, we define its transpose $M^T \in \mathcal{M}_{n \times p}$ by $(M^T)_{i,j} = M_{j,i}$.

Let $A, B, C, D$ be the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -2 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$ 

(i) Identify all of the following products that are defined and give the size of the resulting matrix:


(ii) Compute

$$A(B + C), \quad A(BD), \quad C^T B, \quad (AB)D, \quad D^T D.$$ 

Problem 4.

For any $\theta \in [0, 2\pi)$, let

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in \mathcal{M}_{2 \times 2}.$$ 

(i) Check that $R(\theta)$ is the matrix of a rotation in $\mathbb{R}^2$ around the origin by an angle $\theta$, with respect to the canonical basis $\{(\begin{pmatrix} 1 \\ 0 \end{pmatrix}), (\begin{pmatrix} 0 \\ 1 \end{pmatrix})\}$.

(ii) Use the fact that the composition of a rotation by $\theta$ followed by a rotation by $\omega$ is a rotation by $\theta + \omega$ to derive the trigonometric identities for $\sin(\theta + \omega)$ and $\cos(\theta + \omega)$.

Hint: The composition of maps is given by the product of the corresponding matrices.

(iii) Find the matrix inverse of $R(\theta)$. 