Problem 1. Prove Proposition 3 of the lecture.

Problem 2. Let $V = \mathbb{R}^2$, and $x = \left( \frac{x_1}{x_2} \right), y = \left( \frac{y_1}{y_2} \right)$. Prove that $\{x, y\}$ is linearly independent if and only if $x_1y_2 - x_2y_1 \neq 0$.

Problem 3. Let $V$ be a real vector space and let $S_1, S_2$ be two subsets of $V$ such that $S_1 \subset S_2$. Prove that
(i) if $S_1$ is linearly dependent, then $S_2$ is linearly dependent,
(ii) if $S_2$ is linearly independent, then $S_1$ is linearly independent.

Problem 4. Let $V_1, V_2$ be two subspaces of a real vector space $V$. Define
\[ V_1 + V_2 = \{ v_1 + v_2 : v_1 \in V_1, v_2 \in V_2 \}. \]
If, moreover, $V_1 \cap V_2 = \{0\}$, one writes $V_1 \oplus V_2$ and call the resulting space the direct sum of $V_1$ and $V_2$.
(i) Prove that $V_1 + V_2$ is a vector space.
(ii) Prove $V = V_1 \oplus V_2$ if and only if any vector $x \in V$ has a unique representation as $x = v_1 + v_2$, with $v_1 \in V_1, v_2 \in V_2$.
(iii) Conclude that if $V_1, V_2$ are finite dimensional and $V = V_1 \oplus V_2$, then $V$ is finite dimensional with $\dim(V) = \dim(V_1) + \dim(V_2)$
(iv) Consider the case $V = \mathbb{R}^{n+m}$. Provide a natural direct sum decomposition of $\mathbb{R}^{n+m}$ as $\mathbb{R}^n \oplus \mathbb{R}^m$.

Problem 5. Let $n \in \mathbb{N}$ and let $\mathbb{P}_n$ be the space of polynomials of degree at most $n$
\[ p(t) = a_n t^n + \cdots + a_1 t + a_0 = \sum_{j=0}^{n} a_j t^j, \]
where the coefficients $a_j \in \mathbb{R}$ may be zero. Note that two polynomials are equal if and only if their coefficients are pairwise equal.
Let also $\mathbb{P} = \bigcup_{j \in \mathbb{N}} \mathbb{P}_j$ be the set of polynomials of an arbitrary degree. Note that every $\mathbb{P}_n$ as well as $\mathbb{P}$ are vector spaces (check!).
(i) Show that the dimension of $\mathbb{P}_n$ is $n + 1$.
(ii) Conclude that $\mathbb{P}$ is infinite dimensional.
(iii) Let $n = 2$. Show that the following Hermite polynomials
\[ H_0(t) = 1, \quad H_1(t) = 2t, \quad H_2(t) = 4t^2 - 2, \]
form a basis of $\mathbb{P}_2$.
(iv) What are the coordinates of $H_0, H_1, H_2$ in the basis $\{H_0, H_1, H_2\}$? And in the basis $\{1, t, t^2\}$?