SHARP BERTINI THEOREM FOR PLANE CURVES OVER FINITE FIELDS

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Abstract. We prove that if $C$ is a reflexive smooth plane curve of degree $d$ defined over a finite field $\mathbb{F}_q$ with $d \leq q + 1$, then there is an $\mathbb{F}_q$-line $L$ that intersects $C$ transversely. We also prove the same result for non-reflexive curves of degree $p + 1$ and $2p + 1$ where $q = p^r$.

1. Introduction

A classical theorem of Bertini states that if $X$ is a smooth quasi-projective variety in $\mathbb{P}^n$ defined over an infinite field $k$, then a general hyperplane section of $X$ is smooth. Specializing to the case when $C \subseteq \mathbb{P}^2$ is a smooth plane curve, it follows that there exists a line $L$ (defined over $k$) such that $L$ intersects $C$ transversely, meaning that $C \cap L$ consists of $d$ distinct geometric points where $d = \deg(C)$. But when $k = \mathbb{F}_q$ is a finite field, it is possible to have a smooth plane curve $C \subseteq \mathbb{P}^2$ such that every line $L$ defined over $\mathbb{F}_q$ is tangent to the curve $C$ (see Example 2.A below). Moreover, Poonen’s Bertini Theorem [Poo04, Theorem 1.2] guarantees that such smooth curves, where all the $\mathbb{F}_q$-lines are tangent, do exist in every sufficiently large degree (see Example 2.B below). With a view toward an effective version of Poonen’s theorem, one can ask the following:

Question 1.1. Suppose $C \subseteq \mathbb{P}^2$ is a smooth plane curve defined over $\mathbb{F}_q$. Let $d = \deg(C)$. What conditions on $q$ and $d$ will ensure that there is a line $L \subseteq \mathbb{P}^2$ defined over $\mathbb{F}_q$ such that $L$ meets $C$ transversely?

Let us call $L$ a good line if $L$ meets $C$ transversely. We expect that if $q$ is large with respect to $d$, then good lines will exist. Indeed, if $q \geq d(d-1)$, then the dual curve $C^*$ cannot be space-filling, i.e. $C^*(\mathbb{F}_q) \neq (\mathbb{P}^2)^*(\mathbb{F}_q)$. This is because $\deg(C^*) \leq d(d-1) \leq q$ and a curve of degree of at most $q$ cannot go through all the points of $(\mathbb{P}^2)^*(\mathbb{F}_q)$. Any point in $(\mathbb{P}^2)^*(\mathbb{F}_q) \setminus C^*(\mathbb{F}_q)$ represents a good line $L \subseteq \mathbb{P}^2$ defined over $\mathbb{F}_q$. A generalization of this observation to higher dimensions is proved by Ballico [Bal03, Theorem 1].

In this paper, we improve the quadratic bound $q \geq d(d-1)$ to the linear bound $q \geq d - 1$.

Theorem 1.2. If $C$ is a smooth reflexive plane curve defined over $\mathbb{F}_q$ with $\deg(C) \leq q + 1$, then there is an $\mathbb{F}_q$-line $L$ such that $L$ intersects $C$ transversely.

The theorem is sharp in a sense that the statement cannot be improved to $q \geq d - 2$. There is a counter-example when $q = d - 2$ (see Example 2.A). The “reflexive” assumption on $C$ is same as saying that $C$ has finitely many flex points (see Section 2). As a natural follow-up, we may ask:

Question 1.3. Does Theorem 1.2 hold when $C$ is non-reflexive?

We prove a partial result in this direction:
Theorem 1.4. Let $C$ be a smooth non-reflexive plane curve of degree $p + 1$ or $2p + 1$ defined over $\mathbb{F}_q$ where $q = p^r$ with $r \geq 2$. Then there is an $\mathbb{F}_q$-line $L$ such that $L$ intersects $C$ transversely.

Finally, in the last section of the paper (Section 4), we focus exclusively on Frobenius non-classical curves, which are non-reflexive curves of special kind. As we will see, Question 1.3 in this case is equivalent to a statement about collinear $\mathbb{F}_q$-points on the curve.

Conventions. In order to avoid various pathologies, we will assume throughout the paper that the characteristic of the field is $p > 2$.

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2. Reflexive Curves

In this section we review the theory of reflexive plane curves, and prove Theorem 1.2. If $C$ is a plane curve defined over a field $k$, we can consider the Gauss map $\varphi : C \to (\mathbb{P}^2)^*$ that associates to each smooth point $p$ of $C$ its tangent line. The dual curve $C^*$ is defined to be the closure of $\varphi(C)$ inside $(\mathbb{P}^2)^*$. By looking at the Gauss map for the dual curve, we get $\varphi' : C^* \to C^{**}$. In what follows, we will identify $\mathbb{P}^2$ and $(\mathbb{P}^2)^*$.

Definition 2.1. The curve $C$ is called reflexive if $C = C^{**}$ and $\varphi' \circ \varphi : C \to C^{**}$ is the identity map.

A theorem of Wallace [Wal56] asserts that $C$ is reflexive if and only if $\varphi$ is separable. As a result, all smooth plane curves in characteristic zero are reflexive. Recall that a point $P$ of $C$ is called a flex point if the tangent line at $P$ meets the curve $C$ at $P$ with multiplicity at least 3. When $\text{char}(k) = p > 2$, we have the following characterization: $C$ is reflexive if and only if $C$ has finitely many flex points [Par86, Proposition 1.5].

Before we prove Theorem 1.2, here are some counter-examples of smooth curves $C$ where all the lines defined over $\mathbb{F}_q$ are tangent to $C$ (so that no good line exists).

Example 2.A. Let $C$ be a smooth plane curve with $\text{deg}(C) = q + 2$ such that $\#C(\mathbb{F}_q) = \#\mathbb{P}^2(\mathbb{F}_q)$. Such curves exist, and have been extensively studied by Homma and Kim [HK13]. For such a curve $C$, every $\mathbb{F}_q$-line $L$ intersects $C$ at $q + 2$ points (counted with multiplicity). But $q + 1$ of these points are already accounted by the points of $L(\mathbb{F}_q) = \mathbb{P}^1(\mathbb{F}_q)$. Thus, the residual intersection multiplicity results from $L$ being tangent to $C$ at one of the $\mathbb{F}_q$-points.

Example 2.B. Fix a finite field $\mathbb{F}_q$. Let $\{L_1, ..., L_{q^2 + q + 1}\}$ be all the $\mathbb{F}_q$-lines in the plane. Pick distinct (geometric) points $P_i \in L_i$ for each $i$. The condition that $C$ is tangent to $L_i$ at $P_i$ is a statement about vanishing of the first few coefficients in the Taylor expansion at these finitely many points. By applying Poonen’s Bertini theorem with Taylor conditions [Poo04, Theorem 1.2], there exists some $d_0$ such that for every $d \geq d_0$, there exists a smooth plane curve $C \subseteq \mathbb{P}^2$ of degree $d$ such that $L_i$ is tangent to $C$ at $P_i$. In particular, all $\mathbb{F}_q$-lines $L \subseteq \mathbb{P}^2$ are tangent to $C$. A closer inspection of the proof reveals that the integer $d_0$ is in the order of $q^2$ (essentially because we imposed $q^2 + q + 1$ local conditions).

We will now prove the main theorem of the present paper.
Theorem 1.2. If $C$ is a smooth reflexive plane curve defined over $\mathbb{F}_q$ with $\deg(C) \leq q + 1$, then there is an $\mathbb{F}_q$-line $L$ such that $L$ intersects $C$ transversely.

Proof. Let $\Phi$ be the Frobenius map defined on points by $\Phi([X : Y : Z]) = [X^q : Y^q : Z^q]$. We will write $T_P(C)$ for the tangent line to $C$ at a (geometric) point $P$. Set

$$N = \#{\{P \in C(\mathbb{F}_q) : \Phi(P) \in T_P(C)\}}$$

which is finite because $C$ is reflexive [HV90]. The following inequality is proved in [HKT08, Theorem 8.41]:

$$(*) \quad 2 \cdot \#C(\mathbb{F}_q) + N \leq d(q + d - 1)$$

under the assumption that $C$ has finitely many flex points and that characteristic of the field is $p > 2$. This is the step where we use the hypothesis that $C$ is reflexive.

Assume, to the contrary, that every $\mathbb{F}_q$-line is tangent to the curve $C$ at some (geometric) point. Let us divide these lines into two groups: if $L$ is tangent to $C$ at an $\mathbb{F}_q$-rational point, we will call $L$ a rational tangent. Otherwise, we will call $L$ a special tangent. Since every $\mathbb{F}_q$-line is tangent to $C$, and there are $q^2 + q + 1$ lines defined over $\mathbb{F}_q$, we get

$$\#{\{\text{rational tangents}\}} + \#{\{\text{special tangents}\}} = q^2 + q + 1$$

and

$$\#{\{\text{rational tangents}\}} \leq \#C(\mathbb{F}_q)$$

Now, if $L$ is a special tangent, it is tangent to the curve $C$ at a non-$\mathbb{F}_q$-point $P$. Then $L$ is also tangent to $C$ at $P, \Phi(P), \Phi^2(P), \ldots, \Phi^{e-1}(P)$ where $e = [k(P) : \mathbb{F}_q]$ is the degree of the point $P$. Since $e \geq 2$, the line $L$ contributes at least 2 elements to $N$. As a result,

$$2 \cdot \#{\{\text{special tangents}\}} \leq N$$

Combining all the inequalities above, we obtain that

$$q^2 + q + 1 = \#{\{\text{rational tangents}\}} + \#{\{\text{special tangents}\}}$$

(using $(*)$)

$$\leq \#C(\mathbb{F}_q) + \frac{N}{2} \leq \frac{1}{2}d(q + d - 1)$$

$$\leq \frac{1}{2}(q + 1)(q + (q + 1) - 1) = \frac{1}{2}(q + 1)(2q) = q^2 + q$$

which is a contradiction. $\Box$

When $q = p$ is a prime, every smooth curve of degree at most $p$ is reflexive. Moreover, Pardini [Par86, Proposition 3.7] has shown that every smooth non-reflexive curve of degree $p + 1$ (over any field of characteristic $p$) is projectively equivalent to the curve given by the equation $xyp + yzp + zxp = 0$. For this curve, many good lines exist. For instance, take two $\mathbb{F}_p$-points on the curve, and join them with a line $L$. Then $L$ will intersect $C$ transversely.

Consequently, we deduce the result for all smooth plane curves over $\mathbb{F}_p$ where $p$ is prime.

Corollary 2.2. If $C$ is a smooth plane curve defined over $\mathbb{F}_p$ with $\deg(C) \leq p + 1$ where $p$ is a prime, then there is an $\mathbb{F}_p$-line $L$ such that $L$ intersects $C$ transversely.
3. Non-reflexive curves

In this section, we will restrict attention to non-reflexive curves and prove Theorem 1.4. Let $C \subseteq \mathbb{P}^2$ be a smooth non-reflexive curve defined over $\mathbb{F}_q$ with $q = p^r$ where $r \geq 2$. Pardini [Par86, Corollary 2.4] has shown that $C$ is defined by an equation of the form:

$$a^p x + b^p y + c^p z = 0$$

where $a, b, c \in \mathbb{F}_q[x, y, z]$ are homogeneous polynomials of degree $t \geq 1$. In particular, $\deg(C) = tp + 1$.

We establish a Bertini-type theorem for the case $t = 1$ and $t = 2$.

**Theorem 1.4.** Let $C$ is a smooth non-reflexive plane curve of degree $p + 1$ or $2p + 1$ defined over $\mathbb{F}_q$ where $q = p^r$ with $r \geq 2$. Then there is an $\mathbb{F}_q$-line $L$ such that $L$ intersects $C$ transversely.

**Proof.** When $\deg(C) = p + 1$, then $C$ is projectively equivalent to the curve given by the equation $xy^p + yz^p + zx^p = 0$, for which many good lines $L$ exist (see the discussion before Corollary 2.2). For the rest of the proof, we will assume that $\deg(C) = 2p + 1$. Since $C$ is non-reflexive, by [Par86, Corollary 4.3] the degree of the dual curve is

$$\deg(C^*) = \frac{d(d-1)}{p} = \frac{(2p+1)(2p)}{p} = 4p + 2$$

For $p \geq 5$, we observe that $\deg(C^*) = 4p + 2 \leq p^r \leq q$, so $C^*$ cannot contain all of $(\mathbb{P}^2)^*(\mathbb{F}_q)$, and hence any point $L \in (\mathbb{P}^2)^*(\mathbb{F}_q) \setminus C^*(\mathbb{F}_q)$ will be a desired line that intersects $C$ transversely.

When $p = 3$, the inequality $\deg(C^*) = 4p + 2 = 14 \leq p^r = q$ still holds for $r \geq 3$. The only case that requires a separate analysis is $(p, r) = (3, 2)$, which corresponds to degree $2 \cdot 3 + 1 = 7$ curve defined over $\mathbb{F}_{3^2} = \mathbb{F}_9$. The rest of the proof is devoted to studying this remaining case.

Let $C$ be a smooth non-reflexive curve of degree 7 defined over $\mathbb{F}_9$. Assume, to the contrary, that all the lines defined over $\mathbb{F}_9$ are tangent to $C$. Following the same terminology used in the proof of Theorem 1.2, we call $L$ a **rational tangent** if $L$ is tangent to $C$ at some $\mathbb{F}_9$-point. Otherwise, $L$ is called a **special tangent**. Since $C$ is non-reflexive, each tangent line $L$ must intersect the curve at the tangency point with multiplicity $\geq 3$ (Proposition 1.5 in [Par86]). It follows that:

1. If $L$ is a rational tangent, then $L \cap C$ contains at most five $\mathbb{F}_9$-points.
2. If $L$ is a special tangent, then $L \cap C$ contains a conjugate pair of $\mathbb{F}_{81}$-points and a single $\mathbb{F}_9$-point. In symbols, $L \cap C = \{Q, Q^\sigma, P\}$ where $Q \in \mathbb{P}^2(\mathbb{F}_{81}) \setminus \mathbb{P}^2(\mathbb{F}_9)$ and $P \in \mathbb{P}^2(\mathbb{F}_9)$.

Consider the following incidence correspondence of points and lines,

$$\mathcal{I} = \{(P, L) : L \in (\mathbb{P}^2)^*(\mathbb{F}_9) \text{ and } P \in (C \cap L)(\mathbb{F}_9)\}$$

Each $P \in C(\mathbb{F}_9)$ is contained in $q + 1 = 10$ different $\mathbb{F}_9$-lines. Therefore, $\#\mathcal{I} = \#C(\mathbb{F}_9) \cdot 10$. On the other hand, using (1) and (2) above, each special tangent $L$ contributes 1 point, while each rational tangent $L$ contributes at most 5 points to $\#\mathcal{I}$. Thus, $\#\mathcal{I} \leq S + 5R$ where $S$ and $R$ are the number of special and rational tangents, respectively. We deduce that

$$\#C(\mathbb{F}_9) \cdot 10 \leq S + 5R$$
Since \( \#C(\mathbb{F}_9) \geq R \), we get \( 10R \leq S + 5R \), which implies \( 5R \leq S \). Since \( S + R = 9^2 + 9 + 1 = 91 \), we have \( 5(91 - S) \leq S \), so that \( S \geq \frac{5 \cdot 91}{6} = 75.833... \). Thus, \( S \geq 76 \).

Next, take any rational tangent \( L_0 \). Every special tangent line intersects \( L_0 \) in one of its ten \( \mathbb{F}_9 \)-points. Since \( \frac{S}{10} \geq \frac{76}{10} \geq 7 \), there exists \( P_0 \in L_0(\mathbb{F}_q) \) such that there are at least 8 special tangent lines that pass through \( P_0 \). By looking at the ten \( \mathbb{F}_9 \)-lines passing through \( P_0 \), we can estimate \( \#C(\mathbb{F}_9) \) as follows. Each of the 8 special tangents will contribute at most 1 rational point, while the remaining (at most 2) rational tangents will contribute at most 5 rational points. Thus, one gets \( \#C(\mathbb{F}_9) \leq 8 + 2 \cdot 5 = 18 \). Consider the incidence correspondence:

\[
\mathcal{J} = \{(P, L) : L \text{ is a special tangent and } P \in (C \cap L)(\mathbb{F}_9)\}
\]

By (1) above, every special tangent contains exactly one \( \mathbb{F}_9 \)-point of \( C \), so that \( \#\mathcal{J} = S \). As a result,

\[
S = \#\mathcal{J} = \sum_{P \in C(\mathbb{F}_9)} \#\{\text{special tangents passing through } P\}
\]

Since

\[
\frac{S}{\#C(\mathbb{F}_9)} \geq \frac{76}{18} > 4
\]

there exists a point \( P \in C(\mathbb{F}_9) \) such that at least 5 special tangents pass through \( P \). Consider the corresponding line \( P^* \) in the dual space \((\mathbb{P}^2)^*\), which consists of all lines passing through \( P \). Let us look at the intersection of the line \( P^* \) and the dual curve \( C^* \) inside \((\mathbb{P}^2)^*\). The intersection has all the ten \( \mathbb{F}_9 \)-points of \( P^* \) since all the \( \mathbb{F}_9 \)-lines are tangent to \( C \). However, each of the special tangents is bitangent to \( C \), so it is a node in \( C^* \), and hence will contribute 2 to the intersection. It follows that \( P^* \cap C^* \) has at least 5·2+5 = 15 intersections, contradicting the fact that \( \deg(C^*) = 14 \). \( \square \)

**Remark.** As we saw above, the hardest part of the proof is the case \( p = 3 \). This answers a question of Felipe Voloch, who asked in a private communication, whether or not there exists a transverse line for a degree 7 smooth non-reflexive curve defined over \( \mathbb{F}_9 \). The small primes still persist when we try to extend Theorem 1.3 to non-reflexive curves of degree \( 3p + 1 \). Indeed, if \( C \) is a smooth non-reflexive curve of degree \( 3p + 1 \), then \( \deg(C^*) = \frac{(3p+1)(3p)}{p} = 9p + 3 \leq p^2 \leq q \) for \( p \geq 11 \); the usual argument shows that \( (C^*)(\mathbb{F}_q) \neq (\mathbb{P}^2)^*(\mathbb{F}_q) \), implying that good lines exist for \( p \geq 11 \). However, the main difficulty lies with the primes \( p = 3, 5, 7 \).

4. **Connection to Frobenius non-classical curves**

In this section, we observe the implications of a Bertini-type theorem for a special class of non-reflexive curves, known as Frobenius non-classical curves.

**Definition 4.1.** Let \( C \subseteq \mathbb{P}^2 \) be a smooth plane curve defined over \( \mathbb{F}_q \). Then \( C \) is called **Frobenius non-classical** if \( \Phi(P) \in T_P(C) \) for every \( P \), where \( T_P(C) \) is the tangent line to \( C \) at the point \( P \), and \( \Phi : \mathbb{P}^2 \rightarrow \mathbb{P}^2 \) is the \( q \)-th power Frobenius map.

We should remark that the usual definition of Frobenius non-classical is stated differently (by looking at the order sequence of \( C \)), but the definition given above is equivalent in the case of smooth plane curves [HV90, Proposition 1].
Example. Let $C$ be the curve defined over $\mathbb{F}_{q^2}$ by the equation
\[ x^{q+1} + y^{q+1} + z^{q+1} = 0 \]
It can be checked that $C$ is a smooth Frobenius non-classical curve for $\mathbb{F}_{q^2}$.

If $C$ is a smooth Frobenius non-classical plane curve of degree $d$ defined over $\mathbb{F}_q$ where $q = p^r$, then it is known that $C$ is non-reflexive [HV90, Proposition 1] and $\sqrt{q} + 1 \leq d \leq \frac{q'}{q-1}$ where $q'$ is the generic order of contact of the curve with a tangent line [HV90, Propositions 5 and 6]. In particular, $\deg(C) \leq q - 1$ always holds. So Question 1.3 is equivalent to:

**Question 4.2.** If $C$ is a smooth Frobenius non-classical plane curve defined over $\mathbb{F}_q$, does there exist an $\mathbb{F}_q$-line $L$ such that $L$ intersects $C$ transversely?

The existence of such a line $L$ can be verified for the curve $x^{q+1} + y^{q+1} + z^{q+1} = 0$, and more generally, for the curve given by the equation
\[ x^{q^{n-1}+\cdots+q+1} + y^{q^{n-1}+\cdots+q+1} + z^{q^{n-1}+\cdots+q+1} = 0 \]
where $n \geq 2$. These curves are indeed smooth and Frobenius non-classical with respect to the field $\mathbb{F}_{q^n}$ [HV90, Theorem 2].

If the Question 4.2 has an affirmative answer, then it implies that there is a line $L$ such that $L$ intersects $C$ transversely. Thus, any good (transverse) line $L$ intersects $C$ at $\deg(C)$ distinct $\mathbb{F}_q$-points. This allows us to reformulate Question 4.2 as follows:

**Question 4.3.** If $C$ is a smooth Frobenius non-classical plane curve defined over $\mathbb{F}_q$, then does $C$ have $d = \deg(C)$ many $\mathbb{F}_q$-rational points on a line?

The Question 4.3 is motivated by the fact that Frobenius non-classical curves have many $\mathbb{F}_q$-points. In fact, the $\mathbb{F}_q$-points on these curves have been used in [GPTU02] and [Bor09] to construct certain complete arcs in the plane. Moreover, the following theorem due to Hefez and Voloch [HV90, Theorem 1] gives the exact number of $\mathbb{F}_q$-points on any smooth Frobenius non-classical plane curve:

**Theorem 4.4.** (Hefez-Voloch) If $C \subseteq \mathbb{P}^2$ is a smooth Frobenius non-classical curve of degree $d$ defined over $\mathbb{F}_q$, then
\[ \#C(\mathbb{F}_q) = d(q - d + 2) \]

We can apply Theorem 4.4 directly to get an estimate on the number of collinear points of $C$. Consider the incidence correspondence $\{(P, L) : L \in (\mathbb{P}^2)^*(\mathbb{F}_q) \text{ and } P \in (L \cap C)(\mathbb{F}_q)\}$. Since each $\mathbb{F}_q$-point $P$ is contained in $q + 1$ lines,
\[ \#C(\mathbb{F}_q)(q + 1) = \sum_{P \in C(\mathbb{F}_q)} (q + 1) = \sum_L \#(L \cap C)(\mathbb{F}_q) \]
The sum on the right runs over all $q^2 + q + 1$ lines. Thus, an $\mathbb{F}_q$-line on average contains
\[ \frac{\#C(\mathbb{F}_q)(q + 1)}{q^2 + q + 1} = \frac{d(q - d + 2)(q + 1)}{q^2 + q + 1} > \frac{d(q - d + 2)}{q + 1} > d \left( 1 - \frac{d}{q + 1} \right) \]
$\mathbb{F}_q$-points of $C$. As $q$ gets larger, this number approaches $d$. This heuristic suggests that Question 4.3 may have an affirmative answer.
REFERENCES


