A BERTINI TYPE THEOREM FOR PENCILS OVER FINITE FIELDS

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Abstract. We study the question of finding smooth hyperplane sections to a pencil of hypersurfaces over finite fields.

1. Introduction

Given a smooth projective variety $X \subset \mathbb{P}^n$ over the complex numbers, the classical Bertini theorem asserts the existence of a hyperplane $H$ such that $X \cap H$ is smooth. The statement remains valid over an arbitrary infinite field $k$. For example, every smooth $\mathbb{Q}$-variety admits a smooth $\mathbb{Q}$-hyperplane section. However, if $k = \mathbb{F}_q$ is a finite field, there are counter-examples to the statement. The following example is due to Nick Katz [Kat99]. Consider the surface $S \subset \mathbb{P}^3_{\mathbb{F}_q}$ defined by

$$X^qY - XY^q + Z^qW - ZW^q = 0$$

One can check that each $\mathbb{F}_q$-hyperplane $H \subset \mathbb{P}^3$ is tangent to the surface $S$, and so $S \cap H$ is singular for every choice of $H$ in this case [ADL19, Example 3.4].

If the field $\mathbb{F}_q$ has sufficiently large cardinality with respect to the degree of $X$, then we still expect to find smooth hyperplane sections. A theorem of Ballico [Bal03] shows that for $q \geq d(d-1)^{n-1}$, any smooth hypersurface $X \subset \mathbb{P}^n$ of degree $d$ admits an $\mathbb{F}_q$-hyperplane $H$ such that $X \cap H$ is smooth. When $X$ is a plane curve, a sharper bound of $q \geq d-1$ has been obtained under a stronger hypothesis of reflexivity [Asg19].

We restrict our attention to the case of hypersurfaces. If $X \subset \mathbb{P}^n$ is a hypersurface, we say that a given hyperplane $H$ is transverse to $X$ if $X \cap H$ is smooth.

In this paper, we study a pencil of hypersurfaces defined over $\mathbb{F}_q$ and ask for an $\mathbb{F}_q$-hyperplane which is simultaneously transverse to all the members of the pencil defined over $\mathbb{F}_q$. We begin by taking two different hypersurfaces $X_1 = \{F = 0\}$ and $X_2 = \{G = 0\}$ of the same degree, and considering the $\mathbb{F}_q$-members of the pencil generated by $X_1$ and $X_2$. In other words, we are considering the $q+1$ hypersurfaces,

$$X_{[s:t]} = \{sF + tG = 0\}$$

where $[s : t] \in \mathbb{P}^1(\mathbb{F}_q)$. The main question can be phrased as follows:

**Question 1.1.** Suppose that each member of the pencil spanned by $X_1$ and $X_2$ admits a transverse hyperplane over $\mathbb{F}_q$. Provided that $q$ is sufficiently large with respect to $d$, can we find an $\mathbb{F}_q$-hyperplane $H$ such that $H$ is simultaneously transverse to $X_{[s:t]}$ for each $[s : t] \in \mathbb{P}^1(\mathbb{F}_q)$?

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The case $d = 1$ is clear, because we can just pick $H$ to be any hyperplane that is not in the pencil, and any two distinct hyperplanes intersect transversely. We assume $d > 1$ throughout the paper. In a similar vein with Question 1.1, one may be inclined to ask for the existence of an $\mathbb{F}_q$-hyperplane $H$ such that $H$ is transverse to all the $\mathbb{F}_q$-members of a given pencil. However, this cannot be attained because any hyperplane $H$ must intersect some members of the pencil non-transversely. This is proved in Lemma 3.1.

Our main result asserts that the answer to Question 1.1 is positive if we allow a base extension.

**Theorem 1.2.** Suppose that $X_1, X_2 \subset \mathbb{P}^n$ are two hypersurfaces of degree $d$ defined over a finite field $\mathbb{F}_q$ intersecting properly, and assume that each member of the pencil spanned by $X_1$ and $X_2$ admits a transverse hyperplane over $\mathbb{F}_q$. Furthermore, assume that the pencil has at least one smooth member defined over $\mathbb{F}_q$. Then there exists an integer $s \geq 1$ such that for all positive integers $k$ with $s | k$, the following conclusion holds: for all sufficiently large $q$ with respect to $d$, there exists an $\mathbb{F}_q^k$-hyperplane $H$ such that $H$ is transverse to $X_{[s,t]}$ for each $[s : t] \in \mathbb{P}^1(\mathbb{F}_q^k)$.

**Remark 1.3.** The hypothesis that a pencil has at least one smooth member defined over $\mathbb{F}_q$ is fairly mild. Indeed, a pencil can be viewed as a $\mathbb{P}^1$ inside the parameter space of all hypersurfaces of degree $d$ in $\mathbb{P}^n$. The condition that the pencil admits a smooth member is equivalent to the statement that the corresponding $\mathbb{P}^1$ is not contained inside the discriminant hypersurface $D_{d,n}$, which parametrizes singular hypersurfaces of degree $d$ in $\mathbb{P}^n$. A generically chosen line is not contained inside $D_{d,n}$, and so a generic pencil contains a smooth member.

**Remark 1.4.** According to our definition, a hyperplane $H$ is said to be transverse to $X$ if $H$ provides a smooth hyperplane section of $X$. This condition automatically implies that $H \notin X^*$ where $X^*$ is the dual hypersurface parametrizing tangent hyperplanes to $X$. More precisely, $X^*$ is the closure of the image of the Gauss map of $X$. However, the converse implication is not true. For example a line $L$ passing through the singularity of an irreducible nodal cubic $C$ is not transverse according to our definition, but still satisfies $L \notin C^*$. Some authors, such as [Bal03], defines $H$ to be transverse when the weaker condition $H \notin X^*$ is satisfied. Note that if $X$ is smooth, then $H \notin X^*$ if and only if $X \cap H$ is smooth. Thus, for smooth hypersurfaces, these two definitions of “transverse hyperplane” coincide.

We sketch here the plan for our paper. In Section 2 we discuss our Question 1.1 in the context of plane curves. Then we prove Theorem 1.2 in Section 3. Finally, we conclude our paper by a brief discussion of whether we need to consider a base extension from $\mathbb{F}_q$ to $\mathbb{F}_q^s$, as in the conclusion of Theorem 1.2: in particular, we prove in Proposition 3.3 that for a pencil of reduced plane conics (with at least one smooth conic in the $\mathbb{F}_q^s$-pencil), there always exists a common transverse line to each element of the $\mathbb{F}_q^s$-pencil (as long as $q \geq 16$).

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2. Plane curves

In this Section, we discuss more broadly Question 1.1 in the context of plane curves. In particular, we show (see Proposition 2.3) that given any $N$ reduced
plane curves of degree $d$, there exists a common $\mathbb{F}_q$-line transverse to each one of these $N$ curves, as long as $q \geq 2Nd(d - 1)$. Therefore, it makes sense to consider our Question 1.1 in which we search for a common $\mathbb{F}_q$-line transverse to each curve in a given set of $q + 1$ curves. On the other hand, we show in Example 2.6 that there exists a set of $q + 1$ smooth plane curves with the property that no $\mathbb{F}_q$-line is simultaneously transverse to each curve in our set. Hence, this suggests even more the setup considered in Question 1.1 in which we consider a pencil of plane curves (or more generally of hypersurfaces in $\mathbb{P}^n$).

The setup for this Section is to have two plane curves $C_1 = \{F = 0\}$ and $C_2 = \{G = 0\}$ in $\mathbb{P}^2$ defined over $\mathbb{F}_q$. The polynomials $F, G \in \mathbb{F}_q[x, y, z]$ are homogenous of degree $d$, and we assume that $C_1 \cap C_2$ is finite, i.e. the curves $C_1$ and $C_2$ do not share any components. We consider the pencil of plane curves,

$$C_{[s:t]} = \{sF + tG = 0\}$$

We are interested to find a line $L \subset \mathbb{P}^2$ defined over $\mathbb{F}_q$ such that $L$ is simultaneously transverse to the $q + 1$ members $C_{[s:t]}$ as $[s : t]$ varies in $\mathbb{P}^1(\mathbb{F}_q)$. Note that a line $L \subset \mathbb{P}^2$ is transverse to a curve $C \subset \mathbb{P}^2$ if and only if $L \cap C$ consists of $d = \deg(C)$ distinct points (over $\mathbb{F}_q$).

We need the following result on the number of $\mathbb{F}_q$-points to an arbitrary plane curve which is used in the proof of Proposition 2.2.

**Lemma 2.1.** Suppose $X \subset \mathbb{P}^2$ is a plane curve of degree $d$ defined over $\mathbb{F}_q$. Then the number of $\mathbb{F}_q$-points of $X$ can be bounded by:

$$\#X(\mathbb{F}_q) \leq dq + 1$$

The equality occurs if $X$ is a union of $d$ lines, each defined over $\mathbb{F}_q$, passing through a common $\mathbb{F}_q$-point $P_0$.

**Proof.** Note that if $d \geq q + 1$, then $dq + 1 \geq q^2 + q + 1 = \mathbb{F}^2(\mathbb{F}_q)$, and the claim is trivially true. Thus, we may assume that $d < q + 1$. First, we prove the result in the special case when $X$ has no $\mathbb{F}_q$-linear component. In this case, we prove a slightly stronger bound, namely $\#X(\mathbb{F}_q) \leq dq$. Consider the finite set,

$$\mathcal{I} = \{(P, L) : P \in (X \cap L)(\mathbb{F}_q) \text{ and } L \text{ is an } \mathbb{F}_q \text{-line}\}.$$ 

Given that each $P \in X(\mathbb{F}_q)$ is contained in exactly $q + 1$ lines defined over $\mathbb{F}_q$, we get that $\#\mathcal{I} = (q + 1) \cdot \#X(\mathbb{F}_q)$. On the other hand, using the assumption that $X$ contains no $\mathbb{F}_q$-line as a component, we deduce $L \cap X$ consists of at most $d$ $\mathbb{F}_q$-points by Bezout’s theorem. Since the number of $\mathbb{F}_q$-lines is $q^2 + q + 1$, we obtain,

$$\#\mathcal{I} \leq (q^2 + q + 1)d$$

Combining the two inequalities, we get,

$$(q + 1) \cdot \#X(\mathbb{F}_q) \leq (q^2 + q + 1)d \Rightarrow \#X(\mathbb{F}_q) \leq \left( q + \frac{1}{q + 1} \right) d < q d + 1$$

where in the last step we used $d < q + 1$. Thus, $\#X(\mathbb{F}_q) \leq q d$ for every plane curve $X$ which does not contain an $\mathbb{F}_q$-line as a component.

Now, suppose that $X$ contains an $\mathbb{F}_q$-line as a component. We induct on the degree of $X$ in this case. We write $X = L_0 \cup Y$ where $L_0$ is an $\mathbb{F}_q$-line and $Y$ is a curve of degree $d - 1$. If $Y$ does not contain an $\mathbb{F}_q$-line, then

$$\#X(\mathbb{F}_q) \leq \#L(\mathbb{F}_q) + \#Y(\mathbb{F}_q) \leq q + 1 + (d - 1)q = dq + 1$$
as desired. If $Y$ has an $\mathbb{F}_q$-line $L_1$, then by induction, $\# Y(\mathbb{F}_q) \leq (d-1)q + 1$ but the point $P := L_0 \cap L_1$ is counted twice, so

$$\# X(\mathbb{F}_q) \leq \# L(\mathbb{F}_q) + \# Y(\mathbb{F}_q) - 1 \leq q + 1 + ((d-1)q + 1) - 1 = dq + 1$$

which completes the proof. \hfill $\square$

We note that Lemma 2.1 is covered by a result of Serre [Ser91] who proved a similar upper bound on the number of $\mathbb{F}_q$-points for an arbitrary projective hypersurface in $\mathbb{P}^n$. Serre's result was generalized to all projective varieties by [Cou16].

**Proposition 2.2.** Let $C \subset P^2$ be a reduced plane curve of degree $d$ defined over $\mathbb{F}_q$. If $q \geq 2d(d-1)$, then there exists a transverse $\mathbb{F}_q$-line to $C$.

**Proof.** Given a line $L = \{ax + by + cz = 0\} \subset P^2$, we will show that the condition that $L$ is not transverse to $C = \{F = 0\}$ can be expressed in terms of vanishing of a certain discriminant. Indeed, we can solve for the intersection points $C \cap L$ by substituting $z = -(a/c)x - (b/c)y$ into the equation of $F(x,y,z) = 0$ to obtain $F(x,y,-(a/c)x - (b/c)y) = 0$. After homogenizing (which takes care of the possibility that $c$ could be 0 in the above expression), the equation represents vanishing of a binary form $B_L(x,y)$ of degree $d$ in variables $x$ and $y$ with coefficients that are homogenous in variables $a, b, c$ with degree $d$. The line $L$ is non-transverse to $C$ if this binary form $B_L$ has a repeated root on $\mathbb{P}^1$, i.e. the discriminant of $B_L$ vanishes. Since $\text{disc}(B_L)$ has degree $2d - 2$ in the coefficients of the binary form, and the coefficients themselves are degree $d$ in variables $a, b, c$, we can view

$$\text{disc}(B_L) \in \mathbb{F}_q[a,b,c]$$

as a homogenous form $H$ of degree $(2d-2)d = 2d(d-1)$ in variables $a, b, c$. By viewing a particular line $L$ as a point $[p : q : r] \in (\mathbb{P}^2)^*$ in the dual space, we deduce that $L$ is tangent to $C$ if and only if the point $[p : q : r]$ lies on the plane curve $D = \{H = 0\}$. In particular,

$$\# \{L \in (\mathbb{P}^2)^*(\mathbb{F}_q) \mid L \text{ is a line not transverse to } C\} \leq \# D(\mathbb{F}_q)$$

Since $D$ is a plane curve of degree $2d(d-1)$, the number of $\mathbb{F}_q$-points of $D$ can be bounded by $2d(d-1)q + 1$ by Lemma 2.1. Since the total number of $\mathbb{F}_q$-lines in $\mathbb{P}^2$ is $q^2 + q + 1$, we will obtain a transverse $\mathbb{F}_q$-line to $C$ provided that

$$q^2 + q + 1 > 2d(d-1)q + 1$$

This last inequality is equivalent to $q + 1 > 2d(d-1)$, that is, $q \geq 2d(d-1)$. \hfill $\square$

Using the same idea as in the previous proposition, we obtain:

**Proposition 2.3.** Let $C_1, C_2, \ldots, C_N$ be $N$ reduced plane curves of degree $d > 1$ in $\mathbb{P}^2$ defined over $\mathbb{F}_q$. If $q \geq 2Nd(d-1)$, then there exists a common $\mathbb{F}_q$-line which is simultaneously transverse to $C_i$ for each $1 \leq i \leq N$.

**Proof.** As in the proof of the previous proposition, we obtain that the number of non-transverse $\mathbb{F}_q$-lines to $C_i$ is at most $2d(d-1)q + 1$. Thus, the number of lines that are non-transverse to at least one of the curves $C_1, C_2, \ldots, C_N$ is at most $N \cdot (2d(d-1)q + 1)$. So, we will obtain a common transverse $\mathbb{F}_q$-line to all $C_i$ if

$$q^2 + q + 1 > N \cdot (2d(d-1)q + 1)$$
This inequality will be satisfied for \( q \geq 2Nd(d - 1) \) according to the following computation.
\[
q^2 + q + 1 = q(q + 1) + 1 \geq q(2Nd(d - 1) + 1) + 1 \\
= 2Nd(d - 1)q + q + 1 > 2Nd(d - 1)q + N = N \cdot (2d(d - 1)q + 1)
\]
where in the last inequality we used the fact that \( q + 1 > N \) which is valid under the assumption \( q \geq 2d(d - 1)N \).

However, if the number of curves depend also on \( q \), then the existence of a simultaneous transverse \( \mathbb{F}_q \)-line is not guaranteed.

**Proposition 2.4.** For each \( d \geq 2 \), there exist \( q + 1 \) plane curves \( C_1, C_2, ..., C_{q+1} \) of degree \( d \) such that there is no \( \mathbb{F}_q \)-line which is transverse to each \( C_i \).

**Proof.** Fix an \( \mathbb{F}_q \)-line \( L_0 \) in \( \mathbb{P}^2 \). After enumerating the \( q+1 \) \( \mathbb{F}_q \)-points \( P_1, P_2, ..., P_{q+1} \) on \( L_0 = \mathbb{P}^1 \), construct the curve \( C_i \) such that \( C_i \) is any given degree \( d \) curve that is singular at the point \( P_i \). The resulting collection of curves \( C_1, ..., C_{q+1} \) satisfy the conclusion of the claim. Indeed, each \( \mathbb{F}_q \)-line \( L \) meets \( L_0 \) at a unique point \( P_i \in L_0 \) (depending on \( L \)), and so \( L \) passes through the singular point of \( C_i \), implying that \( L \) is not transverse to \( C_i \). Thus, no \( \mathbb{F}_q \)-line \( L \) can be simultaneously transverse to all the \( q + 1 \) curves \( C_1, C_2, ..., C_{q+1} \).

It would be more satisfying to have examples of smooth curves satisfying the conclusion of Proposition 2.4. We conjecture that such a collection of \( q + 1 \) curves exist.

**Conjecture 2.5.** For each \( d \geq 2 \), there exist \( q + 1 \) smooth curves \( C_1, C_2, ..., C_{q+1} \) in \( \mathbb{P}^2 \) of degree \( d \) such that there is no \( \mathbb{F}_q \)-line which is transverse to each \( C_i \).

We can prove the conjecture in the special case when \( d = 2 \).

**Example 2.6.** Suppose that the characteristic of the field is \( p > 2 \). We want to construct \( q + 1 \) smooth conics \( C_1, ..., C_{q+1} \) such that each \( \mathbb{F}_q \)-line \( L \) in \( \mathbb{P}^2 \) is tangent to at least one of \( C_i \). The set of tangent lines to a given smooth conic \( C \) is parametrized by the dual curve \( C^* \) which also has degree \( d(d - 1) = 2 \). The condition that no \( \mathbb{F}_q \)-line is transverse to all of \( C_1, ..., C_{q+1} \) can be translated into the statement that the \( \mathbb{F}_q \)-points of the corresponding dual curves \( C_1^*, ..., C_{q+1}^* \) fill up all the \( \mathbb{F}_q \)-points of \( (\mathbb{P}^2)^* \).

Motivated by the observation above, we proceed to construct \( q + 1 \) smooth conics \( D_1, D_2, ..., D_{q+1} \) such that
\[
\bigcup_{i=1}^{q+1} D_i(\mathbb{F}_q) = \mathbb{P}^2(\mathbb{F}_q)
\]
Consider the collection of 4 points \( \{P_1, P_2, P_3, P_4\} \subset \mathbb{P}^2(\mathbb{F}_q) \) such that \( \{P_1, P_2, P_3\} \) is a \( \text{Gal}(\mathbb{F}_q^d/\mathbb{F}_q) \)-orbit of the point \( P_1 \in \mathbb{P}^2(\mathbb{F}_q^d) \), while \( P_4 \in \mathbb{P}^2(\mathbb{F}_q) \). In other words, if we write \( P_1 = [a : b : c] \in \mathbb{P}^2(\mathbb{F}_q^d) \), then \( P_2 = [a^q : b^q : c^q] \) and \( P_3 = [a^{q^2} : b^{q^2} : c^{q^2}] \).

Furthermore, we can pick the collection \( B := \{P_1, P_2, P_3, P_4\} \) in such a way that no three of \( P_i \) are collinear. The vector space of homogeneous quadratic polynomials in 3 variables passing through \( B \) has dimension \( 6 - 4 = 2 \), and so we get a pencil.
of conics with base locus $B$. If $\{F_1, F_2\}$ is an $\mathbb{F}_q$-basis for this vector space, then we consider the $q + 1$ members of the pencil,

$$D_{[s,t]} := \{sF_1 + tF_2 = 0\}$$

where $[s,t] \in \mathbb{P}^1(\mathbb{F}_q)$. We claim that each $D_{[s,t]}$ is smooth. Indeed, there are only three singular conics (geometrically) in this pencil, and they are union of two lines passing through $B = \{P_1, P_2, P_3, P_4\}$. Using the notation $\mathcal{PQ}$ for the line passing through $P$ and $Q$, these 3 singular conics are:

$$S_1 := P_1P_2 \cup P_3P_4$$
$$S_2 := P_2P_3 \cup P_1P_4$$
$$S_3 := P_1P_3 \cup P_2P_4$$

However, none of the $S_i$ for $1 \leq i \leq 3$ is defined over $\mathbb{F}_q$. In fact, $S_1$ is strictly defined over the field $\mathbb{F}_q^3$, and Frobenius action sends $S_1 \to S_2 \to S_3 \to S_1$, and so $\{S_1, S_2, S_3\}$ is a Galois orbit of the Frobenius. In particular, each $D_{[s,t]}$ is a smooth conic, and together they cover the $\mathbb{F}_q$-points of $\mathbb{P}^2$. Indeed, on one hand, they all pass through $P_4 \in \mathbb{P}^2(\mathbb{F}_q)$; on the other hand, for each $P \in \mathbb{P}^2(\mathbb{F}_q) \setminus \{P_4\}$, the conic $D_{-[G(P),F(P)]}$ passes through $P$. We re-label the elements of the pencil,

$$\{D_{[s,t]} \mid [s, t] \in \mathbb{P}^1(\mathbb{F}_q)\} = \{D_1, D_2, ..., D_{q+1}\}$$

So $D_1, ..., D_{q+1}$ are smooth conics which together cover the set $\mathbb{P}^2(\mathbb{F}_q)$. Finally, we let $C_i = (D_i)^*$ to be the corresponding dual curve for each $1 \leq i \leq q + 1$. By reflexivity, we have $D_i = (C_i)^*$, and so the tangent lines to $C_i$ for $1 \leq i \leq q + 1$ together cover all the $\mathbb{F}_q$-lines of $\mathbb{P}^2$, i.e. the collection of smooth conics $C_1, ..., C_{q+1}$ admit no common transverse $\mathbb{F}_q$-line.

3. Main Result

In order to establish Theorem 1.2, we will need the following lemma.

Lemma 3.1. Consider a pencil of hypersurfaces generated by $X_1$ and $X_2$ in $\mathbb{P}^n$. Given a hyperplane $H \subset \mathbb{P}^n$, either $H$ is non-transverse to every $\mathbb{F}_q$-member of the pencil, or $H$ is non-transverse to $n(d - 1)^{n-1}$ members of the pencil, counted with appropriate multiplicities.

Proof. We have $X_1 = \{F_1 = 0\}$ and $X_2 = \{F_2 = 0\}$ where $F_1, F_2 \in \mathbb{F}_q[x_0, ..., x_n]$ are homogeneous polynomials of degree $d$. By definition, the elements of the pencil are of the form $X_{[s:t]} = \{sF_1 + tF_2 = 0\}$ as $[s : t]$ varies in $\mathbb{P}^1$. Suppose that $H$ is an arbitrary hyperplane in $\mathbb{P}^n$. After a linear change of coordinates, we may assume that $H = \{x_n = 0\}$. We can restrict the original pencil to the hyperplane $H$ to obtain a new pencil whose elements are of the form,

$$\tilde{X}_{[s:t]} = \{sF_1(x_0, x_1, ..., x_{n-1}, 0) + tF_2(x_0, x_1, ..., x_{n-1}, 0) = 0\}$$

which can be viewed as a pencil of hypersurfaces in $\mathbb{P}^{n-1}$. Note that $H$ is transverse to $X_{[s:t]}$ if and only if $\tilde{X}_{[s:t]} = X_{[s:t]} \cap H$ is smooth. Thus, our task has been reduced to understanding how many of $\tilde{X}_{[s:t]}$ are singular. Let $D_{d,n-1}$ be the discriminant hypersurface parametrizing singular hypersurfaces of degree $d$ in $\mathbb{P}^{n-1}$, and $\mathcal{P} \cong \mathbb{P}^1$ be the pencil whose members are $\tilde{X}_{[s:t]}$. Either $\mathcal{P} \subset D_{d,n-1}$ or $\mathcal{P} \not\subset D_{d,n-1}$. In the first case, $H$ is non-transverse to every member $X_{[s:t]}$ of the original pencil. In the second case, the number of the singular members of $\mathcal{P}$ is given by the degree of the
Thus, the discriminant $D_{d,n-1}$, which is $n(d-1)^{n-1}$ according to [EH16, Proposition 7.4]. Thus, $H$ is non-transverse to exactly $n(d-1)^{n-1}$ members of the original pencil, counted with multiplicity. \hfill \Box

We are now ready to present the proof of the main result.

**Proof of Theorem 1.2.** We have a pencil of hypersurfaces generated by $X_1$ and $X_2$ such that the generic fiber of the pencil is smooth. Given $ζ \in \mathbb{P}^1$, we will denote by $X_ζ$ to be the corresponding member of the pencil. Consider the variety,

$$V = \{(H,ζ) \mid H \text{ is not transverse to } X_ζ\} \subset (\mathbb{P}^n)^* \times \mathbb{P}^1$$

We claim that $V$ is a geometrically irreducible variety. Consider the second projection $π_2: V \to \mathbb{P}^1$. Since the generic member of the pencil is smooth, it follows that the generic fiber is irreducible. Indeed, if $X_ζ$ is smooth, then the fiber

$$π_2^{-1}(ζ) = \{H \in (\mathbb{P}^n)^* \mid H \text{ is tangent to } X_ζ\} = (X_ζ)^*$$

is the dual hypersurface, which is geometrically irreducible as it is the closure of the image of the irreducible hypersurface $X_ζ$ under the Gauss map. Since $π_2: V \to \mathbb{P}^1$ has geometrically irreducible fibers over an open set $U \subset \mathbb{P}^1$ and $V$ is equidimensional (in fact, $V$ is a hypersurface because it can be seen as the dual hypersurface of the generic element of the pencil), it follows that $V$ is geometrically irreducible.

Now, we consider the projection $π_1: V \to (\mathbb{P}^n)^*$. Note that $π_1$ is surjective, because any chosen hyperplane is non-transverse to at least one element of the pencil by Lemma 3.1. In fact, Lemma 3.1 shows that a fiber of $π_1$ either consists of $n(d-1)^{n-1}$ points (which is the generic case) or is an entire $\mathbb{P}^1$. Let

$$Z = \{P \in (\mathbb{P}^n)^* \mid π_1^{-1}(P) = \mathbb{P}^1\}$$

consist of those hyperplanes $P$ that are simultaneously non-transverse to all the members of the pencil. In particular, such a hyperplane $P \in X_1^* \cap X_2^*$ for any two smooth members $X_1, X_2$ of the pencil. This shows that $Z \subset X_1^* \cap X_2^*$ and therefore $\dim(Z) \leq n-2$. In particular, $Z$ is a proper Zariski-closed subset in $(\mathbb{P}^n)^*$. Since $V$ is geometrically irreducible, we can apply [PS20, Theorem 1.8] to deduce that the locus

$$M_{bad} = \{\text{hyperplanes } H \subset (\mathbb{P}^n)^* \mid π_1^{-1}(H) \text{ is not geometrically irreducible}\}$$

differs from a proper Zariski-closed subset by at most a constructible set of dimension 1. As a result, $M_{bad} \neq (\mathbb{P}^n)^*$. Thus, there exists a hyperplane $H \subset (\mathbb{P}^n)^*$ such that $H \notin M_{bad}$. Assuming that $q \gg d$, we can choose such $H := \mathbb{P}^{n-1}$ defined over $\mathbb{F}_q$. Thus, we obtain a map $π_1: π_1^{-1}(H) \to H$. We apply [PS20, Theorem 1.8] again to this new morphism, and continue inductively until we find an $\mathbb{F}_q$-line $B = \mathbb{P}^1 \subset \mathbb{P}^{n-1}$ such that $W := π_1^{-1}(B)$ is a geometrically irreducible curve.

Thus, we obtain a finite map $f: W \to B \cong \mathbb{P}^1$ of geometrically irreducible curves; furthermore, its degree is $m := \deg(π_1)$, which is larger than 1 by Lemma 3.1. Note that $B \subset (\mathbb{P}^n)^*$, so a point $P \in B$ will correspond to a hyperplane $P$ in $\mathbb{P}^n$. The fiber $f^{-1}(P)$ above a given point $P \in B$ will be:

$$f^{-1}(P) = \{ζ ∈ \mathbb{P}^1 \mid P \text{ is non-transverse to } X_ζ\}$$

which is a finite set inside $\mathbb{P}^1$. 
Using the formulation above, we observe that a given $\mathbb{F}_q$-hyperplane $P \in B$ is simultaneously transverse to all the $\mathbb{F}_q$-members of the pencil generated by $X_1$ and $X_2$ if and only if the fiber $f^{-1}(P)$ contains no $\mathbb{F}_q$-points of $\mathbb{P}^1$. In order to obtain the existence of such a point $P$, we will apply the Twisting Lemma of Dèbes and Legrand [DL12] to the cover $W/B$ after applying a suitable base extension (i.e., replacing $\mathbb{F}_q$ by $\mathbb{F}_{qs}$ for a suitable positive integer $s$). Note that $f : W \rightarrow B$ is a cover of geometrically irreducible curves; so, there exists a finite extension $\mathbb{F}_{qs} / \mathbb{F}_q$ such that the base extension of the cover $W_{\mathbb{F}_{qs}} / B_{\mathbb{F}_{qs}}$ has a regular Galois cover $Z_{\mathbb{F}_{qs}} / B_{\mathbb{F}_{qs}}$. More explicitly, $\mathbb{F}_{qs}$ is the closure of $\mathbb{F}_q$ inside the function field $Z(\mathbb{F}_q)$. We also note that if we replace $q$ by any power $q^k$, then it is still true that $Z_{\mathbb{F}_{qs}} / B_{\mathbb{F}_{qs}}$ is a regular Galois cover.

Let $G$ be the Galois group of $Z_{\mathbb{F}_{qs}} / B_{\mathbb{F}_{qs}}$; we view $G$ as a subgroup of $S_m$.

We will apply [DL12, Lemma 3.4] to the map $f : W_{\mathbb{F}_{qs}} \rightarrow B_{\mathbb{F}_{qs}}$ in order to obtain a point $P \in B(\mathbb{F}_{qs})$ with the property that no point in $f^{-1}(P)$ is contained in $W(\mathbb{F}_{qs})$.

We need first a cyclic subgroup $H$ of $G$ generated by an element $\sigma \in S_m$ with the property that $\sigma$ fixes no element in $\{1, \ldots, m\}$ (note that $m > 1$). Indeed, for any Galois group $G$ (seen as a subgroup of $S_m$), there exists an element $\sigma \in G$ which has no fixed point in $\{1, \ldots, m\}$ because $G$ is a transitive group, which means that the stabilizers of the elements in $\{1, \ldots, m\}$ are all conjugated and finally, no group is a union of conjugates of a given proper subgroup.

So, we let $H$ be a cyclic subgroup of $G$ generated by an element $\sigma$ which has no fixed points (as above): we let $r$ be the number of all cycles appearing in $\sigma \in S_m$. We consider the étale $\mathbb{F}_{qs}$-algebra $\prod_{\ell=1}^r E_{\ell}$, where the $E_{\ell}$’s are field extensions of $\mathbb{F}_{qs}$ of degrees equal to the orders of the cycles appearing in the permutation $\sigma$. Then we apply [DL12, Lemma 3.4] to the étale algebra $\prod_{\ell=1}^r E_{\ell} / \mathbb{F}_{qs}$ to obtain a point $P \in B(\mathbb{F}_{qs})$ with the property that $f^{-1}(P)$ splits into $r$ Galois orbits of order $[E_{\ell} : \mathbb{F}_{qs}]$; in particular, none of the points in $f^{-1}(P)$ would be contained in $W(\mathbb{F}_{qs})$ since each of these Galois orbits would have cardinality larger than 1 (because $\sigma$ does not have fixed points).

Now, the hypothesis in applying [DL12, Lemma 3.4] is satisfied because (const/comp) condition from [DL12, Section 3.1.1] is automatically satisfied for regular covers. We need to check the following two conditions, namely [DL12, Lemma 3.4, conditions (ii)-1 and (ii)-2]:

1. This condition is automatically satisfied for large $q$, because the Lang-Weil bounds for the number of points of curves defined over finite fields guarantee the existence of many rational points on the corresponding twisted covers of $Z$, which are curves of the same genus as the genus of $Z$ (see also the proof of [DL12, Corollary 4.3]).

2. This condition is satisfied as explained in the discussion regarding cyclic specializations (since our group $H$ is cyclic) on [DL12, p. 153].

Therefore, [DL12, Lemma 3.4] yields the existence of a point $P \in B(\mathbb{F}_{qs})$ such that no point in $f^{-1}(P)$ is contained in $W(\mathbb{F}_{qs})$. This concludes our proof of Theorem 1.2.

\[\square\]

Remark 3.2. The only point in our proof of Theorem 1.2 where we used that $q$ may have to be replaced by $q^a$ is when considering the Galois closure $Z/B$ for the cover $W/B$ since we want that $Z$ be geometrically irreducible (over $\mathbb{F}_{qs}$). Note that
there are covers of degree larger than 1 of geometrically irreducible curves $W/B$ (over $\mathbb{F}_q$) for which each $\mathbb{F}_q$-point of $B$ has a preimage contained in $W(\mathbb{F}_q)$, thus contradicting the conclusion we seek for the strategy of our proof of Theorem 1.2.

Indeed, we let $W = B = \mathbb{P}^1_{\mathbb{F}_q}$ for some prime power $q$ satisfying the congruence equation $q \equiv 2 \pmod{3}$ and then let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be given by $x \mapsto x^3$. Clearly, $f$ induces a permutation of $\mathbb{P}^1_{\mathbb{F}_q}$; so, each point in $B(\mathbb{F}_q)$ has a preimage contained in $W(\mathbb{F}_q)$. On the other hand, the Galois closure of this cover is $Z = \mathbb{P}^1_{\mathbb{F}_q}$, i.e., we need to perform a base extension of our ground field in order for the Galois cover be geometrically irreducible. Once we replace $q$ by $q^2$, then $W_{\mathbb{F}_q}/B_{\mathbb{F}_q}$ is actually a regular Galois cover and then it is true that there exist points $P \in B(\mathbb{F}_q^2)$ such that no point in $f^{-1}(P)$ is contained in $W(\mathbb{F}_q^2)$.

We do not know whether one can choose $s = 1$ in Theorem 1.2 in general, as our proof strategy requires a base extension (see Remark 3.2). Nevertheless, the following result establishes that $s = 1$ works for the case of pencil of plane conics.

**Proposition 3.3.** Suppose that we have a pencil of reduced conics in $\mathbb{P}^2$ defined over $\mathbb{F}_q$ such that the pencil admits at least one smooth member over $\mathbb{F}_q$. Provided that $q \geq 16$, we can find an $\mathbb{F}_q$-line $L$ that is simultaneously transverse to all the conics defined over $\mathbb{F}_q$ in the pencil.

**Proof.** Suppose that $C_1 = \{F_1 = 0\}$ and $C_2 = \{F_2 = 0\}$ are the two conics that generate the pencil.

We start with some general considerations regarding our proof strategy. First, we observe that if $C$ is a non-smooth reduced conic, then it means that $C$ is a union of two lines $L_1 \cup L_2$ (over $\mathbb{F}_q$) and therefore, we have at most $q + 1$ lines defined over $\mathbb{F}_q$ which are non-transverse to $C$ (they would correspond to all the $\mathbb{F}_q$-lines passing through the $\mathbb{F}_q$-point of $L_1 \cap L_2$). Second, we note that if $C$ is any smooth conic defined over $\mathbb{F}_q$, then the only possibility for an $\mathbb{F}_q$-line $L$ be non-transverse to $C$ is for $L$ to be tangent to $C$ at an $\mathbb{F}_q$-point (since otherwise, we would have that $L$ is tangent to $C$ at two $\mathbb{F}_q$-points, contradiction). In particular, if $C$ is a smooth conic which has no $\mathbb{F}_q$-point, then any $\mathbb{F}_q$-line is transverse to $C$. On the other hand, the number of $\mathbb{F}_q$-points on a smooth $\mathbb{F}_q$-conic (which has at least one $\mathbb{F}_q$-point) is $q + 1$ (since then the conic would be isomorphic to $\mathbb{P}^1$ over $\mathbb{F}_q$); furthermore, each such $\mathbb{F}_q$-point has a tangential line defined over $\mathbb{F}_q$. This provides at most $(q + 1) \cdot (q + 1)$ lines defined over $\mathbb{F}_q$, which are non-transverse to at least one element of the given $\mathbb{F}_q$-pencil. This number is an overestimate since there are only $q^2 + q + 1$ lines defined over $\mathbb{F}_q$, and so there is overcounting that needs to be addressed. In order to refine the counting for the number of non-transverse $\mathbb{F}_q$-lines, we need to take into account the fact that a given $\mathbb{F}_q$-line $L$ will be non-transverse to more than one conic.

In the set-up of the proof for the Theorem 1.2, we have the map $\pi_1 : V \to (\mathbb{P}^2)^*$. Given a line $L \in (\mathbb{P}^2)^*$, the fiber $\pi_1^{-1}(L)$ is either a $\mathbb{P}^1$ or consists of 2 conics according to Lemma 3.1. In the first case, the line $L$ is non-transverse to every element of pencil, and in the second case $L$ is non-transverse to exactly 2 conics (counted with multiplicity). In most cases, we see that each non-transverse $\mathbb{F}_q$-line is counted at least twice. However, there is a locus $B \subset (\mathbb{P}^2)^*$ consisting of those lines $L \in (\mathbb{P}^2)^*$ which are tangent to exactly one conic (with multiplicity 2) in the pencil. We claim that $B$ is a plane curve of degree 4.
The variety $V \subset \mathbb{P}^1 \times (\mathbb{P}^2)^*$ can be described as the locus $\{ R(s, t, a, b, c) = 0 \}$ which has bidegree $(2, 2)$, that is, degree 2 in variables $s, t$ and degree 2 in variables $a, b, c$. The two roots $[s : t] \in \mathbb{P}^1$ satisfying $R(s, t, a, b, c) = 0$ exactly correspond to those members of the pencil to which a given line $L = \{ ax + by + cz = 0 \}$ is non-transverse. The condition that these two roots coincide is controlled by the vanishing of the discriminant $D$ of $R(s, t, a, b, c)$ when $R$ is viewed as a homogeneous quadratic polynomial in $s$ and $t$. Note that $D = D(a, b, c)$ is a degree 4 homogeneous polynomial in $a, b, c$. By definition, $B = \{ D = 0 \}$ and so $\deg(B) = 4$.

By Lemma 2.1, we have $\#B(\mathbb{F}_q) \leq 4q + 1$, and so there are at most $4q + 1$ lines over $\mathbb{F}_q$ which are non-transverse to a single conic (with multiplicity 2) in the pencil.

Finally, there are at most three distinct singular conics in a given pencil of conics by [EH16, Proposition 7.4]. Each such conic is a union of two lines, and the only lines that are not transverse are the $\mathbb{F}_q$-lines passing through the singular point. Thus, there are at most $3(q+1)$ non-transverse lines arising from the singular conics in the pencil.

In total, the number of non-transverse $\mathbb{F}_q$-lines to the $\mathbb{F}_q$-members of the pencil is at most $\frac{(q+1)^2}{2} + 4q + 1 + 3(q+1)$. Since the number of $\mathbb{F}_q$-lines is $q^2 + q + 1$, we get a simultaneously transverse $\mathbb{F}_q$-line provided that,

$$q^2 + q + 1 > \frac{(q+1)^2}{2} + 4q + 1 + 3(q+1)$$

The inequality above is equivalent to $q^2 > 14q + 7$ which is true for $q \geq 16$. □

References


