1 Antiderivatives

Question: given \( \frac{dF}{dx} \) find the original function \( F(x) \).

Example 1. If \( F'(x) = x^3 \) then a solution to our problem is \( F(x) = \frac{1}{4}x^4 \).

Definition 1. Given a function \( f(x) \) we may want to find \( F(x) \) where

\[ F'(x) = f(x) \]

\( F(x) \) is an antiderivative of \( f(x) \)

Example 2 (Simple functions). Find one antiderivative of

1. \( f(x) = x^n \) where \( n \neq -1 \).
   An antiderivative is \( F(x) = \frac{x^{n+1}}{n+1} \).
2. \( f(x) = x^{-1} \).
   An antiderivative is \( F(x) = \log x \).
3. \( f(x) = \sec^2 x \).
   An antiderivative is \( F(x) = \tan(x) \).
4. \( f(x) = \sin(2x) \).
   An antiderivative is \( F(x) = -\frac{1}{2} \cos(2x) \).

Definition 2. Given \( f(x) \), the general form of the antiderivative is given by

\[ F(x) + C, \]

where \( F(x) \) is an antiderivative of \( f(x) \) and \( C \) is a constant.

To find antiderivatives we can use differentiation rules

1. \( \bullet \) Differentiation rule:

\[ (C)' = 0 \]
• **Anti-differentiation rule:** “any constant is an antiderivative of 0”

2. • **Differentiation rule:**

\[(Cf)'(x) = Cf'(x), \quad C \text{ a constant}\]

• **Anti-differentiation constant multiple rule:** If \(F(x)\) is the antiderivative of \(f(x)\) then \(CF(x)\) is the antiderivative of \(cf(x)\).

• **Example:** \(2 \sin x\) is the antiderivative of \(2 \cos x\).

3. • **Differentiation rule:**

\[(f(x) + g(x))' = f'(x) + g'(x)\]

• **Anti-differentiations sum rule:** If \(F(x)\) is the antiderivative of \(f(x)\) and \(G(x)\) is the antiderivative of \(G(x)\) then \(F + G\) is the antiderivative of \(f + g\).

• **Example:** The general antiderivative of \(h(x) = e^x + \sin x\) is \(H(x) = e^x - \cos x\).

4. • **Differentiation rule:** Chain rule:

\[(f(g(x)))' = f'(g(x))g(x)\]

• **Anti-differentiations sum rule:** If \(F(x)\) is the antiderivative of \(f(x)\) and \(G(x)\) is the antiderivative of \(G(x)\) then \(F(G(x))\) is the antiderivative of \(f(G(x))g(x)\).

• **Example:** The general antiderivative of \(h(x) = 2xe^x^2\) is

\[H(x) = e^x^2 + C\]

In this example \(H(x) = F(G(x)) + C\) where \(F(x) = e^x\) and \(G(x) = x^2\). We know that \(f(x) = e^x\) and \(g(x) = 2x\), so we can write \(h(x) = f(G(x))g(x)\).

**Example 3.** Let

\[f(x) = 3x^5 - 7x^2 + 2x + 3 + x^{-1} - x^{-2}.\]

Then the general antiderivative is

\[F(x) = \frac{3}{6}x^6 - \frac{7}{3}x^3 + x^2 + 3x + \log(x) + x^{-1}\]

**Exercise 1.** Find the general antiderivative of

\[f(x) = \frac{2x + x^3}{\sqrt{x}}\]

**Answer.** First we simplify

\[
\begin{align*}
  f(x) &= \frac{2x}{x^{1/2}} + \frac{x^3}{x^{1/2}} \\
      &= 2x^{1/2} + x^{5/2}
\end{align*}
\]

Then, the antiderivative of \(f(x)\) is

\[F(x) = 2 \cdot \frac{2}{3}x^{3/2} + 2 \cdot \frac{2}{7}x^{7/2} + C.\]
Example 4. Assume acceleration due to gravity is constant. The hammer is thrown upwards with velocity $2 \text{m/s}$. Find an equation for $s(t)$ the distance of the hammer from the ground.

Solution. We have 

$$a(t) = -g$$

The antiderivative of $a(t)$ is the velocity 

$$v(t) = -gt + C$$

but $v(0) = 2$. Substituting above we get $C = 2$ and hence 

$$v(t) = -gt + 2.$$ 

The antiderivative of $v(t)$ is the 

$$s(t) = -\frac{1}{2}gt^2 + 2t + C$$

We have $s(0)$ then $c = 0$ and finally we get 

$$s(t) = -\frac{1}{2}gt^2 + 2t + 10$$

We can now easily find the highest point the hammer reaches or the time when the hammer hits the ground. Or the position of the hammer at any time.

Exercise 2. You are driving at $100 \text{km/h}$. You see someone selling ice cream $100 \text{m}$ ahead and slam on the breaks. What constant acceleration is needed to stop in time?

2 Tips for final exam

- Examples done in class
- Webwork problems
- Practice problems
- Quizzes
- Past final exams

Observe course outline and learning goals.