1 Optimization

1. Draw a picture
2. Set up equation
3. Reduce to one variable
4. Calculus
5. Reflect: is the answer reasonable? units?

We begin with a no-frills example. We could elaborate with an interesting story, but this example contains the main steps for solving an optimization problem.

**Example 1** (Distance to a curve). Find the point on the curve \( y = \sqrt{x + 4} \) that is closest to the point \((0,0)\)

1. Draw a picture
2. Set up equation
   
   \[
   d = \sqrt{(x - 0)^2 - (y - 0)^2} = \sqrt{x^2 + y^2}
   \]
3. Reduce to one variable
   
   Use \( y = \sqrt{x + 4} \), then
   
   \[
   d(x) = \sqrt{x^2 + (\sqrt{x + 4})^2} = \sqrt{x^2 + x + 4}
   \]

   **Tip: check domain!** In this case \( x \in (-4, \infty) \)
4. Calculus
   
   Use first derivative test
   
   \[
   d'(x) = \frac{1}{2}(x^2 + x + 4)^{-1/2}(2x + 1) = \frac{2x + 1}{2\sqrt{x^2 + x + 4}}
   \]

   Critical points: \( x = -\frac{1}{2} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>((-4, -1/2))</th>
<th>((-1/2, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>-</td>
<td>+</td>
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   So local min at \( x = -1/2 \). Since there is only one critical point and our function goes from decreasing to increasing the point \((-1/2, \sqrt{3.5})\) is a global minimum.
2 Optimization in physical models

2.1 Refraction

Fermat’s principle of least time: a ray of light will travel along a path that minimizes the time taken.

A uniform medium Consider a ray of light traveling by air, which is the path that minimizes the time taken? In this case the speed of light is constant and

\[
\text{shortest time} = \text{shortest path}
\]

Therefore the ray of light will follow a straight line.

A non-uniform medium

Example 2. A ray of light travels from point \( A \) in a medium where the speed of light is \( c_1 \) to a point \( B \) in a second medium where the speed of light is \( c_2 \). Find the path that this ray of light will follow.

Just for fun, assume that the first medium is air and the second medium is jello. The following are some keypoints:

- By Fermat’s principle, nature will optimize the path of the ray of light so it travels as fast as possible.
- Travelling by air is faster, so the ray will travel as much as possible by air. We no longer get a straight line as an answer.

1. Draw a picture
   - Understand the problem
   - Set up coordinates:
     \[
     A = (0, a), \quad P = (0, x_0), \quad B = (d, b)
     \]

2. Set up equation Recall that

\[
\text{Distance} = \text{rate} \cdot \text{time} \quad \Rightarrow \quad \text{Time} = \frac{\text{distance}}{\text{rate}}
\]

The equation for total time is

\[
T = \frac{|AP|}{c_1} + \frac{|PB|}{c_2}
\]

We have

\[
|AP| = \sqrt{a^2 + x^2}, \quad |PB| = \sqrt{b^2 + (d - x)^2}
\]
3. Reduce to one variable and give optimization domain

\[ T(x) = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d-x)^2}}{c_2} \quad x \in [0, d]. \]

4. Calculus The problem is to find the absolute minimum of \( T(x) \) on the closed interval \([0, d]\).

We have the following theorem from CLP (theorem 3.5.17):

**Theorem 1.** Let \( f(x) \) be defined and continuous. If

\[
\lim_{x \to -\infty} f(x) = \infty \quad \lim_{x \to +\infty} f(x) = \infty
\]

Then the global minimum happens either at

- a critical point \((f'(c) = 0)\), or
- a singular point \((f'(c) \text{ does not exist})\).

We have that

Note that \( T(x) \) is defined and differentiable in all the real numbers.

So we can apply the theorem and the minimum happens at a critical point.

\[
T'(x) = \frac{1}{2} \frac{(2x)}{c_1 \sqrt{a^2 + x^2}} - \frac{1}{2} \frac{2(d-x)}{c_2 \sqrt{b^2 + (d-x)^2}} = \frac{x}{c_1 \sqrt{a^2 + x^2}} - \frac{d-x}{c_2 \sqrt{b^2 + (d-x)^2}}
\]

**Careful with the negative sign!**

Can we guarantee that there exits a solution to:

\[
\frac{x}{c_1 \sqrt{a^2 + x^2}} - \frac{d-x}{c_2 \sqrt{b^2 + (d-x)^2}} = 0
\]

Yes! By the Intermediate Value Theorem. We have

\[
T'(0) < 0, \quad T'(d) > 0
\]

So there exists a critical point \( x_0 \).

If we compute the second derivative we can see that \( T' \) is increasing and therefore this \( x_0 \) the only critical point and hence the global minimum. We get that the global minimum is

\[
T(x_0) = \frac{\sqrt{a^2 + x_0^2}}{c_1} + \frac{\sqrt{b^2 + (d-x_0)^2}}{c_2}
\]

where \( x_0 \) is such that \( T'(x_0) = 0 \), i.e.

\[
\frac{x}{c_1 \sqrt{a^2 + x_0^2}} = \frac{d-x}{c_2 \sqrt{b^2 + (d-x_0)^2}}
\]
5. **Endgame**

The Calculus problem is solved, but we still have pending some a physical conclusion.

If we examine the figure:

\[
\sin(\theta_1) = \frac{x_0}{\sqrt{a^2 + x_0^2}} \quad \text{where } \theta_1 \text{ is the angle of incidence.}
\]

\[
\sin(\theta_2) = \frac{d - x}{\sqrt{b^2 + (d - x_0)^2}} \quad \text{where } \theta_2 \text{ is the angle of refraction.}
\]

we get that \(x_0\) is such that

\[
\sin(\theta_1) = \frac{\sin(\theta_2)}{c_1}
\]

Equivalently,

\[
\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{c_1}{c_2}. \quad \text{Snell’s law.}
\]

Why is this important? Beside advancing our understanding of nature, and being able to explain daily life phenomena, we can use our knowledge of “how nature looks like” to animate movies. *Ray tracing* is a technique to render a 3D image and basically it simulates the path of a ray and the effects of its encounters with virtual objects.

**Exercise 1** (Final 2011). If 24\(m^2\) of material is available to make a rectangular storage container with an open top, and if the length of its base is twice the width, find the largest possible volume of the rectangular storage container. Please justify that your answer gives indeed the largest possible volume.

**Answer.**

\[
V = lwh
\]

we get

\[
V = 8w - \frac{2}{3}w^3
\]

where \(w\) is the width.