1 Topics

- L’Hôpital’s rule

2 L’Hôpital

Recall: limits of quotients

Suppose

\[ \lim_{x \to a} f(x) = A, \quad \lim_{x \to a} g(x) = B; \quad A, B \in \mathbb{R} \]

(i) If \( B \neq 0 \), then

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{A}{B} \]

(ii) If \( B = 0 \) and \( f(x)/g(x) > 0 \) near \( x = a \)

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = +\infty \]

(iii) If \( B = 0 \) and \( f(x)/g(x) < 0 \) near \( x = a \)

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = -\infty \]

(iv) If

\[ \lim_{x \to a} f(x) = A, \quad \lim_{x \to a} g(x) = \pm\infty \]

then

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = 0 \]

**Definition 1** (Indeterminate form). \( \lim_{x \to a} \frac{f(x)}{g(x)} \) is called an indeterminate form of type \( \frac{0}{0} \) if

\[ \lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = 0 \]

We can define other types of indeterminate forms
**Definition 2** (Indeterminate form). \( \lim_{x \to a} \frac{f(x)}{g(x)} \) is called an *indeterminate form* of type \( \frac{\infty}{\infty} \) if

\[
\lim_{x \to a} f(x) = \pm \infty, \quad \lim_{x \to a} g(x) = \pm \infty
\]

Similarly, we can define indeterminate forms for

\[
\lim_{x \to a^+} \frac{f(x)}{g(x)}, \quad \lim_{x \to a^-} \frac{f(x)}{g(x)}, \quad \lim_{x \to \infty} \frac{f(x)}{g(x)}, \quad \lim_{x \to -\infty} \frac{f(x)}{g(x)}
\]

How to handle these? At the beginning of the course we used several techniques (mainly by cancellation).

How about this:

\[
\lim_{x \to 0} \frac{\sin(x)}{\log(1 + x)}
\]

Note that as \( x \to 0 \)

- \( \sin(x) \to \sin(0) = 0 \)
- \( \log(1 + x) \to 1 = 0 \)

This seems to require a new technique

\[
\lim_{x \to 0} \frac{\sin(x)}{\log(1 + x)} = \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \lim_{x \to 0} \frac{x}{\log(1 + x)}
\]

\[
= \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \lim_{x \to 0} \frac{x}{\log(1 + x)}
\]

\[
= \frac{\sin'(0)}{\log'(1 + 0)} = \frac{\cos(0)}{1} = 1
\]

This technique can be generalized to the famous L'Hôpital’s rule.

L'Hôpital’s rule simplifies limits of the form

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

under the following conditions.
Theorem 1 (L’Hôpital’s rule). If

(1) the quotient is an indeterminate form
   - \( \lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x) \) for \( \frac{0}{0} \) type
   - \( \lim_{x \to a} f(x) = \pm \infty = \pm \lim_{x \to a} g(x) \) for \( \frac{\infty}{\infty} \) type

(2) \( f(x), g(x) \) are differentiable around \( x = a \) (except possibly at \( x = a \))

(3) \( \lim_{x \to a} \frac{f'(x)}{g'(x)} \) exists or it is \( +\infty, -\infty \)

The same result holds when \( \lim_{x \to a} \) is replaced by

\[ \lim_{x \to \infty}, \lim_{x \to -\infty}, \lim_{x \to a^+}, \lim_{x \to a^-} \]

Example 1. Evaluate

\[ \lim_{x \to 0^+} x \log x \]

Write this as

\[ \lim_{x \to 0^+} \frac{\log x}{1/x} \]

This is an indeterminate of the form \( \frac{\infty}{\infty} \). Applying L’Hôpital we get

\[ \lim_{x \to 0^+} \frac{\log x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0 \]

Example 2. Evaluate

\[ \lim_{x \to 0^+} \frac{e^x}{x} \]

As \( x \to 0 \) we have \( e^x \to 1 > 0 \) while \( \frac{1}{x} \to +\infty \) (we have positive \( x \) here) so

\[ \lim_{x \to 0^+} \frac{e^x}{x} = +\infty. \]

Warning: do not try to apply L’Hopital on this one!

Advice on taking limits

1. First, simplify as much as possible.

2. See whether it is a determinate form \( (0/1, 1/0, 1/\infty, \infty/1) \) or an indeterminate form \( 0/0, \infty/\infty, \infty \cdot \infty, 0 \cdot \infty, \) etc.
3. For a determinate form use usual methods (including the squeeze theorem).

4. For an indeterminate form see whether L'Hopital can be effectively applied.
   - If it seems to work effectively, it will reduce the problem to an easier one. Try that one.
   - If L'Hopital does not seem to reduce the problem to an easier one, then try other methods (including squeeze theorem).

**Exercise 1.** Compute
\[ \lim_{x \to 1} \frac{x^{11} - 2x + 1}{x - 1} \]

**Exercise 2.** Evaluate
\[ \lim_{x \to 0} \frac{\cos x - 1}{x^2} \]

When you apply L'Hopital many times make sure that:

- each step satisfies 0/0 or \( \infty/\infty \) indeterminate type
- at the last step the limit exists (or = \( \infty \), = \( -\infty \))

Once conditions (1) and (2) in L'Hopital's rule are satisfied, do not worry too much about condition (3) (that \( \lim_{x \to a} f'(x)/g'(x) \) exists) since failure of (3), after (1) and (2), is not usual (but there are examples where it happens, e.g. in examples involving infinitely many oscillations.)

**Example 3.**
\[ \lim_{x \to 0^+} x^x \]

Apply logarithm. Then find
\[ \lim_{x \to 0^+} x \ln x \]

We have that \( x \to 0 \) and \( \ln x \to -\infty \) as \( x \to 0^+ \), so we cannot apply limit laws. Instead we write the limit as a quotient:
\[ \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{x} \]

Then \( \ln x \to -\infty \) and \( \frac{1}{x} \to \infty \) as \( x \to 0^+ \) and both \( \ln x \) and \( \frac{1}{x} \) are differentiable for \( x > 0 \). Then we can apply L'Hopital and get
\[ \lim_{x \to 0^+} \frac{\ln x}{x} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{1} = \lim_{x \to 0^+} \frac{-x^2}{x} = \lim_{x \to 0^+} -x = 0. \]

Back to the limit we are interested in:
\[ \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \exp(\ln(x^x)) = \exp \lim_{x \to 0^+} x \ln(x) = \exp(0) = 1. \]
The trick to solve the last example was to apply logarithm. This also helps in the following exercise.

**Exercise 3.** (*) Find

\[
\lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^{\sqrt{x}}
\]