1 Last class

**Theorem 1** (Increasing/decreasing). 1. If \( f'(x) > 0 \) on an interval then \( f \) is increasing on that interval.

2. If \( f'(x) < 0 \) on an interval then \( f \) is decreasing on that interval.

**Corollary 1.** If \( f'' > 0 \) on an interval then \( f' \) is increasing on that interval.

If we apply this corollary to a critical point, we get a test to determine if a function has a local minima or local maximal at a critical point.

**Theorem 2** (Second derivative test). Suppose \( f'' \) is continuous near \( c \) and \( f'(c) = 0 \) then

1. if \( f''(c) > 0 \) then \( f \) has a local minimum at \( c \).

2. if \( f''(c) < 0 \) then \( f \) has a local maximum at \( c \).

Today we go beyond critical points: from the second derivative we will get information about the shape of the graph.

2 Concavity

**Example 1.** Let \( f(x) = x^3 \).

Look a the tangent lines to the graph of \( x^3 \). The tangent lines are

- above the graph for \( x < 0 \)
- below the graph for \( x > 0 \)
**Definition 1** (Concavity).  
1. If \( f \) lies above all its tangents on an interval \( I \) it is concave upward (concave up / CU) on \( I \).  
2. If \( f \) lies below all its tangents on an interval \( I \) it is concave downward (concave down / DU) on \( I \).

Look again to the tangent lines to the graph of \( x^3 \). The tangent lines are

- decreasing for \( x < 0 \). It means that the second derivative should be negative (if it exists).
- increasing for \( x > 0 \). The second derivative should be positive (if it exists).

**Theorem 3.**  
- If \( f'' > 0 \) then \( f' \) is increasing and hence \( f \) is concave up.
- If \( f'' < 0 \) then \( f' \) is decreasing and hence \( f \) is concave down

**Definition 2.** If \( f''(c) = 0 \) and the concavity of \( f \) changes across \( x = c \), then we call \((c, f(c))\) an inflection point.

**Example 2.** In our running example \( f(x) = x^3 \), \( x = 0 \) is an inflection point.

**Example 3** (A zero is not necessarily an inflection point). Consider \( f(x) = x^4 \).

**Example 4** (Concavity may change where the second derivative doesn’t exist). Consider \( f(x) = x^{1/3} \). Concavity changes at \( x = 0 \), but according to our definition this is not an inflection point.

**Exercise 1.** Let \( f(x) = \frac{\log(x)}{\sqrt{x}} \). The second derivative of \( f(x) \) is

\[
f''(x) = \frac{3\log(x) - 8}{4x^{5/2}}
\]

Determine the intervals on which \( f \) is concave up and the intervals on which \( f \) is concave down. Find the coordinates of any inflection points if they exist.

**Answer.** First we note that the domain of definition of \( f \) is the set of positive numbers \( x > 0 \). Moreover the second exists and is continuous for all \((0, \infty)\), therefore the only change in concavity could happen at \( x = 0 \) such that \( f''(x) = 0 \). Since \( 4x^{5/2} > 0 \) for all \( x > 0 \), to solve \( f''(x) = 0 \) it is enough to solve

\[
3\log(x) - 8 = 0
\]

The solution to the equation above is \( x = e^{8/3} \). This means that \( f''(e^{8/3}) \). Now, we need to check the sign of \( f''(x) \) before \( x = e^{8/3} \) and after \( x = e^{8/3} \):
• If \(0 < x < e^{8/3}\) then \(3 \log(x) - 8 < 0\). Recall that \(4x^{5/2} > 0\) for all \(x > 0\), so we get \(f'(x) < 0\).

• If \(x > e^{8/3}\) then \(3 \log(x) - 8 > 0\). We use again that \(4x^{5/2} > 0\) for all \(x > 0\), so we get \(f'(x) > 0\).

The sign chart looks as follows:

<table>
<thead>
<tr>
<th>(x)</th>
<th>((0, e^{8/3}))</th>
<th>((e^{8/3}, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f'')</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

| concavity   | concave down    | concave up              |

In conclusion:

• The graph of \(f\) is concave down on \((0, e^{8/3})\) and concave up on \((e^{8/3}, \infty)\).

• The point \((e^{8/3}, f(e^{8/3}))\) is an inflection point.