1 Today’s topics

1. Mean Value Theorem

2 About definitions

Critical points: different definitions for High School, CLP-I (lecture) and WeBWork. For example, in WeBWork critical numbers contain both critical and singular points.

3 Mean Value Theorem

Goals:

- To understand the statement of the MVT and Rolle’s theorem
- To be able to apply them, e.g. to show results on number of zeros to an equation, or to estimate values of a function using its derivative.

Consider a function $f$ continuous and differentiable in $[a, b]$ with $f(a) = f(b)$ [Graph] Some observations:

- By the Extreme Value Theorem $f$ should have a global max or global min inside $(a, b)$. (it could happen $f(x) = f(a) = f(b)$ for all $x \in [a, b]$ i.e. the function is constant)
- At these maximum or minimum inside $(a, b)$, the derivative is 0. [By Fermat’s theorem]

Theorem 1 (Rolle’s theorem). Suppose

- $f$ is continuous in $[a, b]$
- $f$ is differentiable on $(a, b)$
- $f(a) = f(b)$

Then there exists $c$ in $(a, b)$ such that

$$f'(c) = 0$$
Example 1. The polynomial function
\[ f(x) = \frac{x^3}{3} - 3x \]
is continuous on \([-3, 3]\) and differentiable on \((-3, 3)\). Since \(f(-3) = f(3) = 0\), by Rolle’s Theorem \(f'\) must be zero at least once in \((-3, 3)\).

In a more general situation, \(f(a) \neq f(b)\). If we rotate \(f\) so that \(f(a) = f(b)\), we would expect the same geometric result to hold: there exists a point \(c\) with the same slope as the secant line through \(f(a)\) and \(f(b)\) (In Role’s theorem, the secant through \(f(a)\) and \(f(b)\) has slope 0).

Theorem 2 (Mean Value Theorem (MVT)). Suppose

- \(f\) is continuous in \([a, b]\)
- \(f\) is differentiable on \((a, b)\)

Then, there exists \(c\) in \((a, b)\) such that
\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

Example 2. Lewis Hamilton and Fernando Alonso raced each other and finished in a tie. Explain, using Rolle’s theorem, why this means there must have been a moment in the race when the two racers were driving at exactly the same velocity.

Answer. Let \(H(t)\) and \(A(t)\) the positions of Hamilton and Alonso at time \(t\), respectively. The problem gives us the following
\[ H(0) = A(0) \quad H(t_0) = A(t_0) \]
where \(t_0\) is the time when they both arrived to the finish line. Consider the difference in the position at time \(t\):
\[ D(t) = H(t) - A(t) \]
By Rolle’s theorem, there exists a time \(c\) between 0 and \(t_0\) such that
\[ D'(c) = 0 \quad \Rightarrow \quad H'(c) = A'(c) \]
Therefore, Lewis Hamilton and Fernando Alonso have the same velocity at time \(c\).

Physical interpretation of MVT. If \(f(t)\) indicates the position of a particle then

- \(f'(c)\) is the instantaneous rate of change (e.g. instantaneous velocity).
\[ \frac{f(b) - f(a)}{b - a} \] is the average rate of change (e.g. average velocity).

The Mean Value Theorem says that at some interior point the instantaneous velocity is equal to the average velocity over the whole interval. Average is also called mean, from there the Mean Value Theorem takes its name.

**MVT is the cornerstone of Calculus**

**Corollary 1.** If \( f'(x) = 0 \) for every \( x \) in \((a, b)\) then \( f(x) \) is a constant function \( f(x) = C \).

**Proof.** Take \( x, y \) in \([a, b]\) with \( x < y \). By the Mean Value Theorem

\[ \frac{f(x) - f(y)}{x - y} = f'(z) \]

for some \( x < z < y \). Since \( f' = 0 \) in \((a, b)\), in particular \( f'(z) = 0 \). Therefore \( (f(x) - f(y))/(x - y) = 0 \) and we get

\[ f(x) = f(y), \quad \text{for all } x, y \in [a, b]. \]

This means that \( f(x) \) is a a constant function. \( \square \)

**Corollary 2.** If \( f'(x) = g'(x) \) for every \( x \) in \((a, b)\), then \( f'(x) - g'(x) \) is constant on \((a, b)\). This is, there exists a constant \( C \) such that \( f(x) - g(x) = C \) for every \( x \) in \((a, b)\). Geometrically, \( f \) and \( g \) are parallel.

These corollaries a “kind of obvious”, but this is the importance of MVT: it allows us to prove the obvious.