1 Today’s topics

1. Optimization: Finding maxima and minima Maximum and minimum values

2. MVT

2 Optimization: extreme values

2.1 Last class

Definition 1. Let $f$ be a function with domain $D$.

- Global maximum on $D$ at $c$:

  $$f(x) \leq f(c) \quad \text{for all } x \in D$$

- Global minimum on $D$ at $d$:

  $$f(x) \geq f(d) \quad \text{for all } x \in D$$

Theorem 1 (Extreme value theorem). If $f$ is continuous on the closed interval $[a, b]$ then $f$ has a global maximum value $f(c)$ and a global minimum value $f(d)$ for some $c, d \in [a, b]$.

Last class we discussed the extreme value theorem and why it should be true. We noticed that two conditions are necessary: continuity and a closed interval. But, why should it be true? We haven’t discussed the necessary definitions to give a proof. If you are curious, the proof of the extreme value theorem is based on deep properties of the real numbers. Specifically, supremum property (a.k.a completeness or least-upper-bound property) of $\mathbb{R}$.

Remark Extremal points could be either at interior points or endpoints.

2.2 Local maxima

Let’s consider the behaviour of the derivative around a maximum.

Example A graph with several local maxima and minima.
Let $f(x)$ be a function defined for $a < c < b$

1. A function $f$ has a local maximum at $c$ if
   
   $f(x) \leq f(c)$ for all $x \in (a', b')$
   
   where $(a', b')$ is an open interval around $c$

2. A function $f$ has a local minimum at $d$ if
   
   $f(x) \geq f(d)$ for all $x \in (a', b)$
   
   where $(a', b)$ is an open interval around $d$.

Why to look to local extrema. Calculus has a local flavour. The derivative can see what is happening in a neighborhood around a point, and compare $f(x)$ with close neighbours to the right $f(x + h)$ and to with close neighbours to the left $f(x - h)$.

**Theorem 2** (Fermat). Suppose

- $f$ is differentiable at $x = c$ and
- $f$ has a local maximum / minimum at $c$

then $f'(c) = 0$

Note that this theorem goes in one direction. Consider the following example.
Example 1. Let
\[ f(x) = x^3 \]
Then \( f'(x) = 3x^2 \). We have that \( f'(0) = 0 \), so \( x = 0 \) is a critical point. But \( x = 0 \) is not a maximum and neither a minimum, because \( f(x) = x^3 \) is an increasing function.

Fermat’s theorem gives a criteria to decide if a point is an extremal point when the function is differentiable. We need to take into account where the function is not differentiable. For more cases, that is all what we need.

You can find the proof in CLP-I. Just at the beginning of the Section 3.5.1 (Local and global maxima and minima). Page 273-274.

3 How to find global min/max

The possible points for extreme values (local or global) are

1. interior points where \( f' = 0 \)
2. interior points where \( f \) is not differentiable
3. endpoints of the domain of \( f \)

We have names for this points

Definition 2. Let \( f(x) \) be a function defined on \( c \).

- If \( f'(c) \) exists and \( f'(c) = 0 \) we call \( c \) a critical point of the function.
- If \( f'(c) \) does not exist then we call \( c \) a singular point of the function
**Philosophical remark.** This is where the power of calculus is: it reduces a problem involving infinitely many points (all the points in an interval) to a problem involving only finite number of points (critical points and singular points)

**Theorem 3.** If a function $f(x)$ has a local maximum or local minimum at $x = c$ and if $f'(c)$ exists, then $f'(c) = 0$.

How to find local extrema:

1. Check critical points
2. Check singular points
3. At these points, check weather there is some interval around $x$ where $f(x)$ is no larger than the other numbers or no smaller. Clues:
   - sketch
   - signs of the derivatives on either side of $x$

How to find global extrema:

1. Evaluate $f(c)$ at all critical
2. Evaluate $f(c)$ at all singular points
3. Evaluate $f(a)$ and $f(b)$
4. Choose largest value (or smallest)

### 4 Examples and exercises

**Example 2.** (Easy, but we didn’t cover it in class.) Find the global minimum and maximum of

$$f(x) = \begin{cases} 
10000 & x = 50 \\
x & \text{otherwise.}
\end{cases}$$

in $0 \leq x \leq 100$.

- Critical points: none.
• Singular points: \( x = 50 \).
• Endpoints: \( x = 100 \) and \( x = 0 \).
• Maximum at \( x = 50 \) and minimum at \( x = 0 \).

**Exercise 1.** Find the absolute maximum and minimum values of \( f(x) = |x| \) on the interval \([-3, 5]\).

**Answer.** Note that \( f \) is continuous in the closed interval \([-3, 5]\), so by the extreme value theorem the absolute maximum and minimum values exist. By observation we get that \( f(5) = 5 \) is the maximum value and \( f(0) = 0 \) is the minimum value.

**Exercise 2.** Find all extrema

\[
f(x) = x^{\frac{2}{3}} - x^2 + 1 \text{ on } [-1, 1]
\]

Find global max/min

**Solution.**

• Observe that \( f \) is continuous on \([-1, 1]\).
• Next we look for critical and singular points.

\[
f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - 2x
\]

• **Singular points:** \( f' \) is not differentiable at \( x = 0 \). The only singular point is \( x = 0 \).
• **Critical points:** Solve

\[
f'(x) = 0
\]

This is

\[
\frac{2}{3}x^{-\frac{1}{3}} - 2x = 0
\]

Solving for \( x \):

\[
2x = \frac{2}{3}x^{-\frac{1}{3}}
\]

\[
x \cdot x^{\frac{1}{3}} = \frac{1}{3}x^{-\frac{1}{3}}x^{\frac{1}{3}}
\]

\[
x^{\frac{4}{3}} = \frac{1}{3}
\]

So \( x^{\frac{1}{3}} = \pm (\frac{1}{3})^{\frac{1}{3}} \) Thus the critical points are

\[
x = \pm (\frac{1}{3})^{\frac{3}{4}}
\]

note that these are points in \([-1, 1]\) since \((\frac{1}{3})^{\frac{3}{4}} < 1\).
• **Endpoints:** \( x = 1, -1 \)

• Compare values of \( f \) at these points:
  - \( f(0) = 1 \)
  - \( f \left( \frac{1}{3} \right)^\frac{3}{2} = \left( \frac{1}{3} \right)^\frac{1}{2} - \left( \frac{1}{3} \right)^\frac{3}{2} + 1 \)
  - \( f \left( -\left( \frac{1}{3} \right)^\frac{3}{2} \right) = \left( \frac{1}{3} \right)^\frac{1}{2} - \left( \frac{1}{3} \right)^\frac{3}{2} + 1 \)
  - \( f(-1) = 1 - 1 + 1 = 1 \)
  - \( f(1) = 1^{2/3} - 1^2 + 1 = 1 \)

  Note \( \left( \frac{1}{3} \right)^\frac{1}{2} - \left( \frac{1}{3} \right)^\frac{3}{2} > 0 \) because \( \frac{1}{3} < 1 \), so \( f \left( \pm \left( \frac{1}{3} \right)^\frac{3}{2} \right) > 1 \)

• **Global maxima** at \( x = \pm \left( \frac{1}{3} \right)^\frac{3}{4} \) with value \( \left( \frac{1}{3} \right)^\frac{1}{2} - \left( \frac{1}{3} \right)^\frac{3}{2} + 1 \)

• **Global minima** at \( x = \pm 1, 0 \) with value 1.