1 Today’s topics

1. More on remainders
2. Optimization: Maximum and minimum values

2 Remainders

Let $R_1(x) = f(x) - T_1(x)$ be the remainder. Then there is $c$ between $a$ and $x$ such that

$$R_1(x) = \frac{f''(c)}{2!}(x - a)^2$$

Example 1. Estimate the error in the linear approximation to $(4.1)^{3/2}$.

First, we need to choose our function $f(x)$ and $a$, around which we make our approximation. Set

$$f(x) = x^{3/2}, \quad a = 4$$

To apply the Lagrange Remainder formula we need the second derivative of $f(x)$. We have the following:

$$f'(x) = \frac{3}{2}x^{1/2}, \quad f''(x) = \frac{3}{4}x^{-1/2}$$

Then

$$R_1(x) = \frac{f''(c)}{2!}(x - a)^2 = \frac{3}{4}c^{-1/2}(0.1)^2, \quad \text{for some } 4 < c < 4.1$$

3 Taylor series

We look for transcendental functions like $e^x$, $\sin x$ or $\cos x$. Precisely, because the definition of a transcendental function is one that cannot be written exactly as a polynomial.

The main idea is that a bigger $n$ gives a better expansion, so that the remainder gets smaller and smaller. Equivalently

$$\lim_{n \to \infty} R_n(x) = 0$$

and so

$$\lim_{n \to \infty} T_n(x) = f(x)$$
If $R_n(x) \to 0$, then we can write $f$ as an infinite sum
\[
 f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots \\
= \sum_{k=0}^{\infty} \frac{1}{k!}f^{(k)}(a)(x-a)^k
\]
This is called the Taylor series of $f$ at $a$. More of this in a later course.

4 Optimisation: maximum and minimum values

4.1 Global extreme values

**Definition 1.** Let $f$ be a function with domain $D$.

- $f$ has an **global (absolute) maximum** value on $D$ at a point $c$ if
  \[ f(x) \leq f(c) \quad \text{for all } x \in D \]
- $f$ has an **global (absolute) minimum** value on $D$ at $c$ if
  \[ f(x) \geq f(c) \quad \text{for all } x \in D \]

The global maximum and global minimum of $f$ are called **extreme values**.

**Examples**

1. Consider $y = x^2$ on the following domains
   - (a) $D = (-\infty, \infty)$: no absolute maximum, absolute minimum at $x = 0$
   - (b) $D = [-2, 2]$, absolute maximum at $x = 2$, absolute minimum at $x = 0$
   - (c) $D = (0, 2]$, absolute maximum at $x = 2$, no absolute minimum
   - (d) $D = (0, 2)$, no absolute extrema

2. Now, consider a non-continuous function
   \[
   f(x) = \begin{cases} 
   \frac{1}{2} & \text{if } x = 0 \\
   \frac{1}{2} & \text{if } x = 1 \\
   x & \text{otherwise}
   \end{cases}
   \]
   in the interval $[0, 1]$. No absolute extrema: you can always have $x_1$ such that $f(x_1)$ is very close to 0 -but never 0- and $x_2$ such that $f(x_2)$ is very close to 1 -but is never 1-.
**Question:** under which condition can we guarantee that we always have a global maximum or global minimum? **Hint:** Think on what went wrong for $f$ and $D$ without absolute extrema.

**Theorem 1** (Extreme value theorem). If $f$ is continuous on the closed interval $[a, b]$ then $f$ has a global maximum value $f(c)$ and a global minimum value $f(d)$ for some $c, d \in [a, b]$. 