1 Today’s topics

• Taylor polynomials
• Remainders i.e. how good is our approximation with Taylor polynomials

2 Taylor’s polynomials

**Definition 1** (Taylor polynomial). Let \( a \) be a constant and let \( n \) be a non-negative integer. The \( n \)th degree Taylor polynomial for \( f(x) \) about \( x = a \) is

\[
T_n(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a) \cdot (x - a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a) \cdot (x - a)^n
\]

\[
= \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(a) \cdot (x - a)^k
\]

The special case \( a = 0 \) is called a Maclaurin polynomial

**Equation 1.** The Taylor polynomial for \( f \) about \( a \) is an approximation to \( f(x) \) for \( x \) close to \( a \)

\[
f(x) \approx T_n(x)
\]

**Discussion: why a higher degree makes a better approximation.** Consider polynomial of degree 1, say

\[
p_1(x) = c_1x + c_0
\]

The graphical representation of \( p_1(x) \) is a line and this line has exactly one zero. Now, if we consider a polynomial of degree 2,

\[
p_2(x) = c_2x^2 + c_1x + c_0
\]

then we have at most two roots. There are polynomials of degree 2 with 0, 1 and two roots, but we can choose the coefficients such that we have two roots, e.g.

\[
x^2 - 1 = (x + 1)(x - 1)
\]

You can think that we took the line and “bent” it, so \( p_2(x) \) has two roots. In the same way, polynomials of degree 3 have at most three roots, and with the right coefficients these three roots
are different real numbers
\[ p_3(x) = (x + 1)(x - 1)(x + 2) \]

Now, if we have more freedom to pick the shape of our function, it is reasonable to say that we get a better approximation with Taylor polynomials of higher degree.

In some cases, we don’t need a higher degree. Consider the function
\[ p(x) = x^2 - 1 \]

The linear approximation of \( p(x) \) about 0 is
\[ L(x) = p(0) + p'(0)(x) = -1 \]

The quadratic approximation about 0 is
\[ T_2(x) = \frac{p''(0)}{2} x^2 + p'(0)x + p(0) = x^2 + 1 \]

The Maclaurin polynomial of \( p(x) \) of degree 3 is
\[ T_3(x) = x^2 + 1, \]

and the Maclaurin polynomial of \( p(x) \) of degree 4 is
\[ T_4(x) = x^2 + 1, \]

and the same for all the Maclaurin polynomials of higher degree. This is because the quadratic approximation is as good as it gets... it is the function itself.

Maclaurin polynomials of degree \( n \) of some functions you know and love

- \( e^x \approx 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} \)
- \( \sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} \), where \( 2k + 1 \) is the greatest odd integer less than or equal to \( n \).
- \( \cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{(-1)^k x^{2k}}{(2k)!} \), where \( 2k \) is the greatest even integer less than or equal to \( n \).
- \( \ln(1 - x) \approx -x - \frac{x^2}{2} - \ldots - \frac{x^n}{n} \)
- \( \arctan(x) \approx x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{(-1)^k x^{2k+1}}{2k+1} \), where \( 2k + 1 \) is the greatest odd integer less than or equal to \( n \)
- \( \frac{1}{1-x} \approx 1 + x + x^2 + \ldots + x^n \)

Exercise 1. Find the 1st and 2nd order Taylor expansions of \( x^{3/2} \) about \( x = 4 \) and use them to approximate \( (4.1)^{3/2} \)
Answer. We approximate the function \( f(x) = x^{3/2} \). The first and second derivatives are

\[
\begin{align*}
    f'(x) &= \frac{3}{2}x^{1/2} \\
    f''(x) &= \frac{3}{4}x^{-1/2}
\end{align*}
\]

Evaluated at 4 we get

\[
\begin{align*}
    f(4) &= 4^{3/2} = 8 \\
    f'(4) &= \frac{3}{2}4^{1/2} = 3 \\
    f''(4) &= \frac{3}{4}4^{-1/2} = \frac{3}{8}
\end{align*}
\]

The first order Taylor expansion of \( x^{3/2} \) about \( x = 4 \) is

\[
T_1(x) = f(4) + f'(4)(x - 4) = 8 + 3(x - 4)
\]

The second order Taylor expansion of \( x^{3/2} \) about \( x = 4 \) is

\[
T_2(x) = f(4) + f'(4)(x - 4) + \frac{f''(4)}{2}(x - 4)^2 = 8 + 3(x - 4) + \frac{3}{8}(x - 4)^2
\]

Finally, we use both the first and second Taylor expansions of \( x^{3/2} \) to approximate \((4.1)^{3/2}\). The approximation of first order is

\[
(4.1)^{3/2} \approx T_1(4.1) = 8 + 3(.1) = 8.3
\]

The approximation of second order is

\[
(4.1)^{3/2} \approx T_2(4.1) = 8 + 3(.1) + \frac{3}{8}(.1)^2 = 8.30375
\]

Exercise 2. Find the 2nd order Taylor expansion of \( x^{3/2} + 3x \) about \( x = 4 \).

Answer. This exercise can be solved with the formula for Taylor polynomial. But we don’t have to work harder, just smarter. Recall that the derivative is linear. Let

\[
\begin{align*}
    h(x) &= x^{3/2} + 3x, \\
    f(x) &= x^{3/2} \\
    g(x) &= 3x
\end{align*}
\]

Write \( T^h_2, T^f_2 \) and \( T^g_2 \) be the 2nd order Taylor expansions of \( h(x), f(x) \) and \( g(x) \), respectively. We have that

\[
T^h_2(x) = T^f_2(x) + T^g_2(x) \tag{1}
\]

Indeed, it just follows from linearity of differentiation, since (with \( a = 4 \)):

\[
\begin{align*}
    T^h_2(x) &= h(x) + h'(x)(x - a) + \frac{h''(x)}{2}(x - a)^2 \\
    &= f(x) + g(x) + (f'(x) + g'(x))(x - a) + \frac{f''(x) + g''(x)}{2}(x - a)^2 \\
    &= T^f_2(x) + T^g_2(x).
\end{align*}
\]
Now, given (1) we only need $T^f_2(x)$ and $T^g_2(x)$. From Exercise 2 we have

$$T^f_2(x) = 8 + 3(x - 4) + \frac{3}{8}(x - 4)^2.$$ 

The 2nd order Taylor expansion of $3x$ is easy to get from our definition of Taylor polynomial. Recall that $T^g_2$ is such that $T^g_2(a) = g(a)$, $(T^g_2)'(a) = g'(a)$ and $(T^g_2)''(a) = g''(a)$). This determines a unique polynomial of degree 3, which makes for the best approximation of third degree. But $g$ is a polynomial of first degree, so the best approximation is the function itself:

$$T^g_2(x) = 3x$$

You can apply Definition 1 and it will give you the same. The key point is to realize that there you don’t need to compute anything. Finally, we get the 2nd order Taylor expansion of $x^{3/2} + 3x$ about $x = 4$:

$$T^h_2(x) = 8 + 3(x - 4) + \frac{3}{8}(x - 4)^2 + 3x$$

$$= 20 + 6(x - 4) + \frac{3}{8}(x - 4)^2$$

Again, you can check that applying Definition 1 gives the same answer -but that’s not a surprise! We made a smart use of our Theorems and definitions.

**Exercise 3. (**)** Find the 8th order expansion of $f(x) = e^{x^2} + \cos(5x)$. What is $f^{(6)}(0)$?

### 3 **Taylor’s formula with remainder**

**Question.** How well does the Taylor polynomial of degree $n$ approximate the function $f$?

We look to the difference between $f(x)$ and $T_n(x)$: the *remainder*

$$R_n(x) = f(x) - T_n(x)$$

**Remark** Now we can write an *equality*:

$$f(x) = f(a) + f'(a)(x - a) + \ldots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x)$$

**Theorem 1** (The Lagrange remainder formula). Let $[b, d]$ an interval with $x, a \in [b, c]$. Suppose that $f$ has derivatives of at least order $n + 1$ on $[b, d]$. The remainder for the Taylor polynomial of $f(x)$ about $x$ can be written

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}$$

where $c$ is some number between $x$ and $a$: $a < c < x$. 

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The proof of the Lagrange remainder formula need the Mean Value Theorem (MVT). We will see MVT next week.

**Remark** In equation (2) we know that such $c$ exists, but we do not know its precise value. The key point is to estimate $f^{(n+1)}(c)$ using the fact that $c$ is between $x$ and $a$.

Recall that $T_n$ and $R_n$ depend on our choice of $a$, we could write $R_{n,a}(x) = f(x) - T_{n,a}(x)$. We want to estimate with absolute values:

$$|f(x) - T_n(x)| = |R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} |x - a|^{n+1}$$

**Observation** The error $|R_n(x)|$ gets smaller and smaller as $n$ increases, especially due to the fact that the factorial $(n+1)!$ increases very fast in $n$. 