1 Today’s topics

1. Approximation
2. Taylor polynomials

2 Goal

Consider a polynomial function like
\[ p(x) = x^3 + 9x^2 - 3x + 2 \]
A polynomial is easy to evaluate at any point: we only need arithmetic (\(+\), \(-\), \(\times\),).

Another kind of function, like \(\sin x\) or \(\ln x\) is more difficult to evaluate (or to understand how a calculator evaluates them).

**Question.** Is it possible to approximate a given function by a polynomial?

We want to approximate a function with a more simple one. We will go successive stages. We will approximate the value of a function with

1. a constant
2. a linear approximation
3. a quadratic function
4. a general polynomial

3 Constant approximation

Approximate with a constant function (a polynomial of degree 0). We approximate \(f(x)\) near \(x = a\) by the function \(T(x) = \text{constant}\).

How to make it good? We will choose the constant so that the original function and our approximation agree at \(x = a\).

**Example 1.** We want to estimate \(\sin(0.2)\). We approximate by \(a = 0\) and say
\[ \sin(0.2) \approx \sin(0) = 0 \]
Figure 1: Graph of $f(x) = x^{3/2}$.

**Question.** Why is this bad? The function $\sin(x)$ is increasing, then $\sin(0.2) > \sin(0)$.

**Characteristics of a good approximation**

- accurate
- possible to calculate (add, subtract, multiply, divide integers)

**Key observation.** The tangent line indicates that the function is increasing. Moreover, the tangent line shows if the function is dramatically increasing or just a little bit increasing i.e. how much is the difference between $(9.1)^{3/2}$ and $9^{3/2}$

We can get a better approximation using a tangent line.

### 4 Linear approximation

If a constant approximation is the “zeroth approximation” that is an approximation of “zeroth order”. Following the same pattern, a linear approximation is an approximation of *first order* because we approximate with a polynomial of degree 1.

The *linearization* of $f$ at $x = a$ is the tangent line

$$L(x) = f(a) + f'(a) \cdot (x - a)$$
Notice that
\[ L(a) = f(a) \quad L'(a) = f'(a) \]
Indeed, \( L(x) \) is the unique first degree polynomial that satisfies these two conditions.

**Example 2.** We want to estimate \( \sin(0.2) \).

The zeroth order approximation is \( \sin(0) = 0 \). The answer will be close to this, but we need to make a correction. Let
\[ f(x) = \sin(x) \]
and let’s find its linear approximation at \( x = 0 \).
\[ f(x) = \sin(x) \quad f'(x) = \cos(x) \]
Then
\[ f(x) \approx f(a) + f'(a) \cdot (x - a) \]
Correction of first order

Now, we approximate around 9 to estimate 9.1
\[ f(0 + 0.2) \approx 0 + 1 \cdot (0.2 - 0) \]
\[ = 0.2 \]

We we can make correction of higher order!

A computer gives
\[ \sin(0.2) = 0.19866933079... \]

**Key points** To find a linear approximation of \( f(x) \) at a particular point \( x \):

- pick a point \( a \) near to \( x \)
- \( f(a) \) and \( f'(a) \) are easy to calculate
- approximate
\[ f(x) \approx L(x) = f(a) + f'(a)(x - a) \]

[Worksheet]

**Exercise 1.** Determine what \( f(x) \) and \( a \) should be so that you can approximate the following:

(a) \( \ln(0.9) \)
(b) \( e^{-1/30} \)
(c) \( \sqrt[3]{30} \)
(d) \( (2.01)^6 \)
Exercise 2. We want to estimate \((9.1)^{3/2}\).

The zeroth order approximation is \(9^{3/2} = 27\). The answer will be close to this, but we need to make a correction. Let
\[
f(x) = x^{3/2}
\]
and let’s find its linear approximation at \(x = 9\).
\[
f(x) = x^{3/2} \quad f'(x) = \frac{3}{2} x^{1/2}
\]
Then
\[
f(x) \approx f(a) + f'(a) \cdot (x - a)
\]
Correction of first order

Now, we approximate around 9 to estimate 9.1
\[
f(9 + 0.1) \approx 9^{3/2} + \frac{3}{2} 9^{1/2} \cdot (9.1 - 9)
\]
\[
= 27 + \frac{3}{2} 9^{1/2} \cdot (9.1 - 9)
\]
\[
= 27 + \frac{9}{20}
\]
\[
= 27 + 0.45
\]
\[
= 27.45
\]

We can make correction of higher order! That is what the computer does. It gives
\[
(9.1)^{3/2} = 27.45124770...
\]

Exercise 3. Use a linear approximation to estimate \(e^x\) near \(x = 0\)

5 Quadratic approximation

Example 3. Estimate \(\sin \left( \frac{\pi}{2} + 0.1 \right)\)

Our first attempt is a linear approximation. We have
\[
\sin \left( \frac{\pi}{2} \right) = 1 \quad \sin' \left( \frac{\pi}{2} \right) = \cos \left( \frac{\pi}{2} \right) = 0.
\]
Then, the linear approximation around \(\pi/2\) is just the constant approximation
\[
L(x) = \sin(\pi/2) = 1.
\]
The situation would be better if we can approximate with a parabola (a polynomial of second order)

Let us examine the general situation. We chose the constant approximation \(T(x)\) by requiring
\[
T(a) = f(a).
\]
We improved with a linear approximation by requiring
\[
\begin{align*}
T(a) &= f(a) \\
T'(a) &= f'(a)
\end{align*}
\]
Next, we can make a quadratic approximation by requiring
\[
\begin{align*}
T(a) &= f(a) \\
T'(a) &= f'(a) \\
T''(a) &= f''(a)
\end{align*}
\] (1)
and the quadratic approximation is:
\[
T(x) = c_0 + c_1(x - a) + c_2(x - a)^2,
\]
We have
\[
\begin{align*}
T'(x) &= c_1 + 2c_2(x - a) \\
T''(x) &= 2c_2
\end{align*}
\] ⇒ \[
\begin{align*}
T'(a) &= c_1 \\
T''(a) &= 2c_2
\end{align*}
\]
To satisfy the requirements in (1) we need
\[
\begin{align*}
c_0 &= f(a) \\
c_1 &= f'(a) \\
c_2 &= f''(a)/2
\end{align*}
\]
Therefore
The quadratic approximation (or the second degree Taylor polynomial for \( f \) about \( x = a \)) is
\[
T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2
\]
To finish the example above, the quadratic approximation to \( \sin(x) \) around \( \frac{\pi}{2} \) is

\[
T_2(x) = 1 - \frac{1}{2} \left(x - \frac{\pi}{2}\right)^2
\]

Therefore

\[
\sin \left(\frac{\pi}{2} + 0.1\right) \approx 1 - \frac{1}{2}(0.1)^2
\]

**Example 4.** Approximate \( f(x) = e^x \) close to \( x = 0 \).

We need to know \( f(0), f'(0) \) and \( f''(0) \):

\[
f(x) = e^x \quad f(0) = e^0 = 1
\]

\[
\left\{ \begin{array}{l}
  f(x) = e^x \\
  f'(x) = e^x \\
  f''(x) = e^x
\end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l}
  f(0) = e^0 = 1 \\
  f'(0) = 1 \\
  f''(0) = 1
\end{array} \right.
\]

Substituting these into the equation gives the second degree Taylor polynomial for \( e^x \) about \( x = 0 \):

\[
T_2(x) = 1 + x + \frac{1}{2}x^2
\]

A Taylor approximation about \( x = 0 \) has a special name: “Maclaurin polynomials”.

### 6 Taylor polynomials

We can do a higher degree Maclauring polynomial for \( e^x \) very easily. Again, we need a requirement:

\[
T_n^{(m)}(0) = f^{(m)}(0)
\]

Then

\[
\left\{ \begin{array}{l}
  T_n(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n \\
  T_n'(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1} \\
  T_n''(x) = 2c_2 + 3 \cdot 2c_3x + \cdots + n(n-1)c_nx^{n-2} \\
  T_n^{(3)}(x) = 3 \cdot 2c_3 + \cdots + n(n-1)(n-2)c_nx^{n-3}
\end{array} \right. \\
\Rightarrow \left\{ \begin{array}{l}
  T_n(0) = c_0 \\
  T_n'(0) = c_1 \\
  T_n''(0) = 2c_2 \\
  T_n^{(3)}(0) = 3 \cdot 2c_3
\end{array} \right.
\]

Recall that \( \exp^{(m)}(0) = 1 \). Then

\[
c_0 = 1, \quad c_1 = 1, \quad c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{2 \cdot 3}, \quad c_4 = \frac{1}{2 \cdot 3 \cdot 4}, \quad \ldots, \quad c_n = \frac{1}{n \cdot (n-1) \ldots \cdot 2 \cdot 1} = \frac{1}{n!}
\]
Therefore, the \textit{\textit{\textit{\textit{\textit{n}-th degree Taylor polynomial of } e^x \text{ about } x = 0}} is}

\[ T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \]

Let \( a \) be a constant and let \( n \) be a non-negative integer. The \( n \)-th degree Taylor polynomial for \( f(x) \) about \( x = a \) is

\[ T_n(x) = f(a) + f'(a) + \frac{1}{2}f''(a)(x - a)^2 + \ldots + \frac{1}{n!}f^{(n)}(x - a)^n \]

\[ = \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(x - a)^k \]