1 Review of inverse functions

What is a function? Usually we consider $f : \mathbb{R} \rightarrow \mathbb{R}$.

(a) $F : \text{“People”} \rightarrow \text{“numbers with 8 digits”}$. A function between people and student numbers

(b) $G : \text{“student numbers in Math 100”} \rightarrow \text{“Section”}$. A function between student numbers registered in Math 100 and their section

input $x \rightarrow f$ does “stuff” to $x \rightarrow$ return unique output $y$

We are interested in the inverse operation

take output $y \rightarrow$ do “stuff” to $y \rightarrow$ return original $x$

For which of our examples above can we defined the inverse?

(a) $F$ is invertible
(b) $G$ is non-invertible

Smart observation In order for a function to be invertible different $x$ values cannot map to the same $y$ value

**Definition 1** (One-to-one). A function $f$ is one-to-one (injective) when

\[
\text{if } x_1 \neq x_2 \text{ then } f(x_1) \neq f(x_2)
\]

(Equivalently, the contrapositive states: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$) That is, when it never takes the same $y$ value more than once.

**Example 1** (One-to-one real function). $f(x) = 3x$
Question: is it injective?

Yes. Let’s prove it. If \( f(x) = f(y) \) then \( 3x = 3y \). Dividing by 3 we get \( x = y \). Therefore \( f \) satisfies the definition and it is one-to-one.

In the case of a function \( f : \mathbb{R} \to \mathbb{R} \) we have a geometric test

[Graph \( f(x) = 3x \)]

**Definition 2** (Horizontal line test). A function is one-to-one if and only if no horizontal line \( y = c \) intersects the graph \( y = f(x) \) more than once.

**Example 2.** \( \sin(x) \) is not injective. It fails the horizontal line test [Graph \( \sin(x) \)]

**Definition 3** (Inverse function). Let \( f \) be a one-to-one function with domain \( A \) and range \( B \). Then its inverse function is denoted \( f^{-1} \) and has domain \( B \) and range \( A \). It is defined by

\[
  f^{-1}(y) = x \quad \text{whenever} \quad f(x) = y
\]

for any \( y \in B \).

**Warning:** \( f^{-1}(x) \neq \frac{1}{f(x)} \)

**Example 3.** The inverse function of \( f(x) = \frac{1}{3}x \)

**Step 0** Check \( f \) is one-to-one in its domain

**Step 1** Solve for \( x \)

**Step 2** Interchange \( x \) and \( y \)

**Step 3** Domain \( f^{-1} = \text{Range } f \)

The graphs of \( y = f(x) \) and \( y = f^{-1} \) are symmetric with respect to the identity line \( y = x \)

This is simply because for the inverse function the roles of \( x \) and \( y \) as input and output are interchanged.

1.1 Worksheet

**Exercise 1.** Find the function inverse to \( y = x^7 + 3 \)
**Answer.** To find the inverse we need to: 1. solve for $y$ and 2. exchange $y$ and $x$.

**Step 1: solve for $y$** We have

$$x = (y - 3)^{1/7}$$

**Step 2: exchange $y$ and $x$** We get

$$y = (x - 3)^{1/7}$$

and this is the inverse function.

We didn’t check if the function is actually one-to-one. This is very easy with the following fact:

**Composition of one-to-one functions gives a one-to-one function**

If $f(x) = x^7$ and $g(x) = x + 3$ then $h(x) = x^7 + 3$ is the composition

$$h(x) = g \circ f = g(f(x))$$

Both $f$ and $g$ are one-to-one, therefore the composition $h$ is one-to-one too. From the point of view of composition of functions, we have a second way to get the inverse. The inverse of $f$ is

$$f^{-1}(x) = x^{1/7}$$

and the inverse of $g$ is

$$g^{-1}(x) = x - 3$$

Then the inverse of $h(x) = g(f(x))$ is

$$h^{-1} = f^{-1}(g^{-1}(x)) = f^{-1}(x - 3) = (x - 3)^{1/7}$$

(2)

This is the same than [1]. First we need to undone $g$ (because it is the last function we apply for $h$), and then we undo whatever $f$ did.

A way to check if (2) is right answer is with the following observation:

$$h^{-1}h(x) = h(h^{-1}(x)) = x.$$  

Let us check our answer above. We have:

$$h(h^{-1}(x)) = h((x - 3)^{1/7}) = [(x - 3)^{1/7}]^{7} - 3 = x - 3 + 3 = x; a$$

and

$$h^{-1}(h(x)) = h^{-1}(x^7 + 3) = ([x^7 + 3] - 3)^{1/7} = (x^7)^{1/7} = x.$$  

**Theorem 1.** Suppose $f$ and $g$ are one-to-one. Then $f \circ g(x) = f(g(x))$ is one-to-one

**Exercise 2.** Let $f(x) = \sqrt{x - 1}$
(a) Find the domain of \( f \)
(b) Show that \( f \) is one-to-one
(c) Find \( f^{-1} \) (in the form \( x = g(y) \))
(d) Find \( \frac{dy}{dx}, \frac{dx}{dy} \) and calculate their product

**Answer.** (a) The domain of \( f \) is \( \{x \in \mathbb{R} | x \geq 1 \} \)

(b) To show one-to-one property, we check

\[
 f(x_1) = f(x_2) \Rightarrow x_1 = x_2
\]

Suppose

\[
 \sqrt{x_1 - 1} = \sqrt{x_2 - 1}.
\]

The square root is an invertible function in its domain of definition, then \( x_1 - 1 = x_2 - 1 \) and therefore \( x_1 = x_2 \). This proves \( f(x) = \sqrt{x - 1} \) is invertible.

**Alternative** We discussed an alternative way to prove that the function is invertible. The key observation is that \( \sqrt{x^2 - 1} \) is increasing. A complete justification needs tools from following weeks, but it is a nice way to approach the problem. The derivative of \( f(x) = \sqrt{x - 1} \) is

\[
 \frac{d}{dx} f(x) = \frac{1}{2\sqrt{x-1}} > 0
\]

Since the derivatives is positive in all the domain, then \( f \) is increasing. Now, take \( x \) and \( y \) with \( x \neq y \).

Without loss of generality

\[
 x < y
\]

(someone should be the smaller, we call it \( x \)). Since the function is increasing, it follows that

\[
 f(x) < f(y)
\]

In particular

\[
 f(x) \neq f(y).
\]

Therefore \( f \) is a one-to-one function, since it satisfies the definition.

(c) **Step 1: a quick calculation** Write

\[
 y = \sqrt{x - 1}
\]

When \( x \geq 1 \), there is a unique \( y \) such that

\[
 y^2 = x - 1
\]

thus

\[
 x = y^2 + 1
\]

**Step 2: identify the domain of \( f^{-1} \)** Note:

Domain of \( f^{-1} = \) range of \( f(x) = \sqrt{x - 1} \)
Note $\sqrt{x-1} \geq 0$ Then:

- the range of $f$ is $\{y \mid y \geq 0\}$
- the domain of $f^{-1}$ is $\{y \mid y \geq 0\}$

Finally, the complete answer is

$$f^{-1}(y) = y^2 + 1, \quad \text{for } y \geq 0$$

**Step 3: check the answer** Let’s check $f(f^{-1}(y))$. We have:

$$f(f^{-1}(y)) = f(y^2 + 1) = \sqrt{y^2 + 1} - 1 = \sqrt{y^2} = |y|$$

Since the domain of $f^{-1}$ are the non-negative numbers, we have $|y| = y$ and hence

$$f(f^{-1}(y)) = y.$$

**Exercise 3.** Does $y = x^2$ have an inverse?

**Answer.** No. Graphically, it fails the horizontal line test. It is easy to see that it fails the definition: Consider 1 and $-1$. We have that $1 \neq -1$ but $1^2 = (-1)^2 = 1$.

## 2 Logarithms

**Example 4.** Consider the exponential function

$$y = e^x$$

The function is

- Domain: $\mathbb{R}$ (equivalently $-\infty < x < \infty$)
- Range: $y > 0$
- Injective (equivalently, one-to-one)

Its inverse is

$$y = \ln x$$

The logarithm is

- Domain $x > 0$
- Range: $-\infty < y < \infty$

Graph: $y = e^x$ and $y = \ln x$ are symmetric about $y = x$.

**Inverse exponential function**

\[1\] For this exercise, we denote the input for $f^{-1}$ by $y$. Recall that usually we denote the input by $x$. 

5
**Definition 4.** The logarithm with base $q$ is defined by

$$y = \log_q(x) \iff x = q^y$$

**Properties of logarithms**

**Theorem 2 (Logarithm rules).** Let $A$ and $B$ positive numbers and let $n$ be any real number.

1. $\ln(A \cdot B) = \ln(A) + \ln(B)$
2. $\ln(A/B) = \ln A - \ln B$
3. $\ln(A^n) = n \ln(A)$

**Proof.**

(a) We want: $\ln(A \cdot B) = \ln(A) + \ln(B)$ Write

$$\ln(A \cdot B) = \ln(e^{\ln A} e^{\ln B})$$

We apply laws for exponents

$$\ln(e^{\ln A} e^{\ln B}) = \ln(e^{\ln A + \ln B}) = \ln(A) + \ln(B)$$

(b) Want: $\ln(A/B) = \ln A - \ln B$ Write in terms of the exponential and apply exponent laws:

$$\ln(A/B) = \ln\left(\frac{e^{\ln A}}{e^{\ln B}}\right) = \ln\left(e^{\ln A - \ln B}\right) = \ln A - \ln B$$

(c) Want: $\ln(A^n) = n \ln(A)$ Same strategy than above: write in terms of the exponential and apply exponent laws.

$$\ln(A^n) = \ln((e^{\ln A})^n) = \ln(e^{n \ln A}) = n \ln A.$$

The last property today is the change of base

We have

$$b^{\log_b(a)} = a$$

Now we want to write this in terms of the natural logarithm. Apply $\ln$:
\[ \ln(b^\log_b(a)) = \ln(a) \]

By the logarithm laws, we can take the exponent out as a multiplication Then

\[ \log_b(a) \ln(b) = \ln(a) \]

Therefore

Change of base - to natural base:

\[ \log_b(a) = \frac{\ln a}{\ln b} \]

In general:

For positive \( a, b \) and \( c \):

\[ \log_b(a) = \frac{\log_c(a)}{\log_c(b)} \]

2.1 Worksheet

Exercise 4.

1. \( \log(e^{10}) = \)

2. \( \log(2^{100}) = \) (in terms of \( \log 2 \))

Exercise 5. Simplify into a single logarithm:

\[ f(x) = \ln \left( \frac{10}{x^2} \right) + 2 \ln x + \ln(10 + x) \]

Exercise 6. A variant on Moore’s law states that computing power doubles every 18 months. Suppose computer today can do \( N_0 \) operations per second.

(a) Write a formula for the future power of computers:

Answer. Computers \( t \) years from now will be able to do \( N(t) \) operations per second where

\[ N(t) = 2^{t/1.5}N_0 \]

(b) A computing task would take 10 years for today’s computers. Suppose we wait 3 years and then start the computation. When will we have the answer?

Answer. In 3 years,

\[ N(3) = 2^{3/1.5}N_0 = 2^2N_0 = 4N_0. \]

Then computers will be 4 times more powerful. If the task takes 10 years to complete if we do it today, in 3 years it would take
\[ \frac{10}{4} = 2.5 \text{ years} \]

(c) At what time will computers be powerful enough to compute the task in 6 months?

**Answer.** If computer can complete a task in 6 months, it needs to be \( x \) times more powerful, where

\[ \frac{10}{x} = \frac{1}{2} \quad \Rightarrow \quad x = 20 \]

Then, we need computers 20 times more powerful than today. We want to find \( t \) such that

\[ N(t) = 20N_0 \quad \Rightarrow \quad 2^{t/1.5} = 20. \]

We can solve applying the logarithm base 2 (inverse function of \( 2^x \)). We get

\[ \frac{t}{1.5} = \log_2(20) \quad \Rightarrow \quad t = (1.5)\log_2(20) = \left(\frac{3}{2}\right)\log_2(20). \]

Simplifying with the logarithmic laws:

\[ t = \log_2(20^{\frac{3}{2}}) \approx 6.48289 \text{ years} \]