Motivation and the limit of a function

This lecture is based on our textbook CLP-I, sections 1.2 and 1.3.

Definitions covered in class

1. Limit of a function
2. Left- and right- handed limits

1 Motivation

The following are some real-life phenomena:

- Movement
  - Consider an object falling from the top of the tower of Pisa.
  - We relate time and distance by a function
    \[ s : \mathbb{R} \to \mathbb{R} \]
    \[ t \mapsto d \]
- Population growth
  - Similarly, time and population size are related by a function.

**Question.** Can you think of other (physical, biological, sociological...) phenomena where two quantities are related by a function?

About differential calculus:

- *Object of Study:* functions
- *Main Question:* how does a function change?
- *Main Idea:* approximation.

**Example 1** (Velocity problem). Back to a hammer falling from the tower of Pisa.
- Time \( t \) (unit = seconds)
- \( s(t) \) is the distance from the top / the hammer has fallen (unit = meters)
- Model
  \[
  s(t) = \frac{9.8}{2} t^2 = 4.9t^2.
  \]

**Question.** How fast is the hammer falling after 1 second?

More precisely, what is the (instantaneous) velocity at \( t = 1 \)?

**Note.** This is an example of the Main Question. Let’s use the Main Idea: approximation!

We begin with average velocity.

\[
v_a(1.1) = \text{“average velocity between 1 and 1.1”}
= \frac{\text{change in position}}{\text{change in time}}
= \frac{s(1.1) - s(1)}{1.1 - 1}
= \frac{4.9(1.1)^2 - 4.9}{0.1}
= \frac{4.9 \times 0.21}{0.1}
= 10.29 \text{m/s}
\]

Now, let’s look how \( v_a(x) \) changes when \( x \) is closer and closer to 1:

\[
\begin{align*}
v_a(1.1) &= 10.29 \\
v_a(1.01) &= 9.849 \\
v_a(1.001) &= 9.8049 \\
v_a(1.0001) &= 9.80049
\end{align*}
\]

**Question.** What should be the instantaneous velocity?

We get closer to 9.8. Therefore, the instantaneous velocity is

\[
v(1) = 9.8 \text{m/s}
\]

On Example 1, we arrived to the solution with an approximation:

\[
v(1) = \lim_{h \to 0} v_a(h) = \lim_{h \to 0} \frac{s(1 + h) - s(h)}{h},
\]

\( ^{1}\text{Thank you, Galileo!} \)
2 The limit of a function

2.1 Definition and first examples

Let \( f : \mathbb{R} \to \mathbb{R} \) be a function.

**Definition 1.** We will often write

\[
\lim_{x \to a} f(x) = L ; \text{ or } \quad f(x) \to L \text{ as } x \to a
\]

which means the “limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \).”

**Definition 2** (Limit of a function). We write

\[
\lim_{x \to a} f(x) = L
\]

if the value of the function \( f(x) \) is sure to be arbitrarily close to \( L \) whenever the value of \( x \) is close enough to \( a \) without being exactly \( a \).

**Question** (Example 0). If \( f(x) = x - 2 \), what is \( \lim_{x \to 2} f(x) = ? \).

**Graph:** (take a pencil)

Of course, the answer is \( \lim_{x \to 2} f(x) = 0 \).

**Example 2** (A simple limit).

\[
f(x) = \begin{cases} 
2x & x < 3 \\
17 & x = 3 \\
2x & x > 3 
\end{cases} \tag{2}
\]
We have 

\[ f(3) = 17 \quad \lim_{x \to 3} f(x) = ? \]

**Question.** What is \( \lim_{x \to 3} f(x) \)?

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
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<tbody>
<tr>
<td>2.9</td>
<td>5.8</td>
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<tr>
<td>2.99</td>
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<td>2.999</td>
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<td>6.02</td>
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<td>3.001</td>
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We conclude 

\[ \lim_{x \to 3} f(x) = 6. \]

**Example 3.** Consider 

\[ f(x) = \frac{x - 2}{x^2 + x - 6}. \]

Determine 

\[ \lim_{x \to 2} f(x) \]

**Question.** What should we do?

*First try.* Evaluation gives \( \frac{0}{0} \).

This expression is NOT DEFINED\(^2\) and cannot be an answer.

*Second try.* Let’s use some algebra. We have the following factorization:

\[ \frac{x - 2}{x^2 + x - 6} = \frac{x - 2}{(x - 2)(x + 3)} = \frac{1}{x + 3}. \]

\(^2\)In this case, not defined means that there is no sensible way to make sense of this expression. It’s not 1, neither 0, and definitely not \( \infty \).
Then
\[
\lim_{x \to 2} \frac{x - 2}{x^2 + x - 6} = \lim_{x \to 2} \frac{1}{x - 2 + 3}
\]
This last limit is easy to determine, as in Example 0:
\[
\lim_{x \to 2} \frac{1}{x + 3} = \frac{1}{5}
\]
The conclusion is
\[
\lim_{x \to 2} \frac{x - 2}{x^2 + x - 6} = \frac{1}{5}.
\] (3)

2.2 One sided limits

Example 4. Consider the function

\[
f(x) = \begin{cases} 
  x & x < 2 \\ 
  -1 & x = 2 \\ 
  x + 3 & x + 3x > 2
\end{cases}
\]

Graph:(take a pencil)

Let us plug in numbers close to 2

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
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<tbody>
<tr>
<td>1.9</td>
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<td>2.001</td>
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In this case
\[
\lim_{x \to 2} f(x) \text{ does not exist}
\]
However, we can talk about “one-sided limits”

**Definition 3** (One-sided limits. Definition 1.3.7 in CLP). We write

$$\lim_{x \to a^-} f(x) = K$$

when the value of $f(x)$ gets closer and closer to $K$ when $x < a$ and $x$ moves closer and closer to $a$. Since the $x$-values are always less than $a$, we say that $x$ approaches $a$ from below. This is also often called the *left-hand limit* since the $x$-values lie to the left of $a$ on a sketch of the graph. We similarly write

$$\lim_{x \to a^+} f(x) = K$$

when the value of $f(x)$ gets closer and closer to $L$ when $x > a$ and $x$ moves closer and closer to $a$. For similar reasons we say that $x$ approaches $a$ from above, and sometimes refer to this as the *right-hand limit*.

**Note.** The following are other notations.

$$\lim_{x \to a^+} f(x) = \lim_{x \downarrow a} f(x) = \lim_{x \nearrow a} f(x) = L \quad \text{right-hand limit}$$

$$\lim_{x \to a^-} f(x) = \lim_{x \uparrow a} f(x) = \lim_{x \searrow a} f(x) = L \quad \text{right-hand limit}$$

In Example 4, we have

$$\lim_{x \to 2^-} f(x) = 2 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 5$$

**Theorem 1** (Theorem 1.3.8 in CLP-1). We have that

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

### 2.3 When the limit does not exist

**Example 5** (A bad example). Consider the function

$$f(x) = \sin \left( \frac{\pi}{x} \right)$$

Observe that

$$\lim_{x \to 0} f(x) \text{ DNE (does not exist)}$$

This is clear when we draw the graph, since any real number $L$ does not meet all the requirements from Definition 2.
Graph:

(take a pencil. Hint: How does the graph of $sin(y)$ look like?)