1. Section 13.1, Problem 2

**Solution:** First observe that function \( f : (0, \infty) \to \mathbb{R} \) defined by \( f(x) = \ln(x) \) is bijective. (Proof: one can prove that a function is bijective by showing that the inverse function exists. Here the inverse function is \( f^{-1}(x) = e^x \). Make sure you understand how this works!)

Then the formula \( f(x) = e^x + \sqrt{2} \) gives a bijective function from \( \mathbb{R} \) to \((\sqrt{2}, \infty)\).

Here is a formal proof that it is bijective. We need to prove that it is both surjective and injective.

**Surjective:** we need to show that for every \( y \in (\sqrt{2}, \infty) \) there exists \( x \in \mathbb{R} \) such that \( e^x + \sqrt{2} = y \).

Since \( y > \sqrt{2} \), we have that \( y - \sqrt{2} > 0 \), hence \( x = \ln(y - \sqrt{2}) \) is well defined, and for this value of \( x \) we have \( f(x) = 2 \), which proves surjectivity.

**Injectivity:** We need to prove that if \( x_1 \neq x_2 \) then \( f(x_1) \neq f(x_2) \). We have:

\[
e^{x_1} + \sqrt{2} = e^{x_2} + \sqrt{2} \iff e^{x_1} = e^{x_2} \iff x_1 = x_2,
\]

where the last biconditional holds since the exponential is strictly increasing.

2. Section 13.1, Problem 8

**Solution:** Note that \( \sin(x) = 1 \) if and only if \( x = \pi/2 + 2\pi k \) for some \( k \in \mathbb{Z} \). Thus \( S = \{ \pi/2 + 2\pi k : k \in \mathbb{Z} \} \). Now let \( f : \mathbb{Z} \to S \) be defined by the formula \( f(k) = \pi/2 + 2\pi k \). Proving that it is bijective (similarly to the previous problem) is an easy exercise.

3. Section 13.1, Problem 14

**Solution:** Let \( A = \{ (n, m) \in \mathbb{N} \times \mathbb{N} : n \leq m \} \). Define \( f : \mathbb{N} \times \mathbb{N} \to A \) by \( f(n, m) = (n, m + n - 1) \). Let us prove that this is indeed a function from \( \mathbb{N} \times \mathbb{N} \) to \( A \) and it is bijective. First, we observe that indeed, the image of \( f \) lies in \( A \): since \( m \geq 1 \), we have that \( n \leq n + m - 1 \). Thus we really have constructed a function from \( \mathbb{N} \times \mathbb{N} \) to \( A \). Second, we need to prove that this function is bijective.

**Surjectivity:** let \( (a, b) \in A \), which means, \( b \geq a \). We need to prove that there exists \( (n, m) \in \mathbb{N} \times \mathbb{N} \) such that \( f(n, m) = (a, b) \). This would mean: \( n = a \) and \( m + n - 1 = b \). So let \( n = a \), let \( m = b + 1 - a \). Since \( b \geq a \), we have that \( m > 0 \), so \( m \in \mathbb{N} \). Thus for every \( (a, b) \in A \) we have found \( (n, m) \in \mathbb{N} \times \mathbb{N} \) such that \( f(n, m) = (a, b) \), and surjectivity is proved.
Injectivity: We need to prove that if \( f(n_1, m_1) = f(n_2, m_2) \) then \( n_1 = n_2 \) and \( m_1 = m_2 \). Direct proof: suppose \((n_1, m_1)\) and \((n_2, m_2)\) are two pairs of natural numbers such that \( f(n_1, m_1) = f(n_2, m_2) \). Looking at the first coordinate of \( f(n, m) \), we get that \( n_1 = n_2 \) immediately. Now look at the second coordinate: we have that \( m_1 + n_1 - 1 = m_2 + n_2 - 1 \), and we already know that \( n_1 = n_2 \). Then it follows that \( m_1 = m_2 \), and we are done.

4. Let \( A_1, A_2, B_1, B_2 \) be non-empty sets such that \( |A_i| = |B_i| \) for \( i = 1, 2 \). Prove that
   (a) \( |A_1 \times A_2| = |B_1 \times B_2| \).
   (b) If \( A_1 \cap A_2 = B_1 \cap B_2 = \emptyset \), then \( |A_1 \cup A_2| = |B_1 \cup B_2| \).

Remember — the sets may or may not be finite. This also applies to the remaining questions below.

Solution: (a)

- Since \( |A_i| = |B_i| \) there exist bijections \( f_i : A_i \to B_i \), \( i = 1, 2 \).
- Define \( h : A_1 \times A_2 \to B_1 \times B_2 \) by \( h(a_1, a_2) = (f_1(a_1), f_2(a_2)) \). We must show it is a bijection.
- Injection. Let \((a_1, a_2), (a_1', a_2') \in A_1 \times A_2 \). Assume that \( h(a_1, a_2) = h(a_1', a_2') \). By our definition of \( h \) we know that \( f_1(a_1) = f_1(a_1') \) and so \( a_1 = a_1' \). Similarly, \( f_2(a_2) = f_2(a_2') \) and so \( a_2 = a_2' \). Thus \((a_1, a_2) = (a_1', a_2') \) and so \( h \) is injective.
- Surjection. Let \((b_1, b_2) \in B_1 \times B_2 \). Since \( f_1 \) and \( b_2 \) are surjective there are \( a_1 \in A_1 \) and \( a_2 \in A_2 \) so that \( f(a_i) = b_i \), for \( i = 1, 2 \). Now \((a_1, a_2) \in A_1 \times A_2 \) and \( h(a_1, a_2) = (b_1, b_2) \). Thus \( h \) is surjective.
- Since \( h \) is injective and surjective, it is bijective and the two sets have the same cardinality.

(b) Since \( |A_1| = |B_1| \) and \( |A_2| = |B_2| \), there exist bijective functions \( f_1 : A_1 \to B_1 \) and \( f_2 : A_2 \to B_2 \). We need to construct a bijective function \( h : A_1 \cup A_2 \to B_1 \cup B_2 \). Let us define it piece-wise:

\[
h(x) = \begin{cases} 
  f_1(x) & \text{if } x \in A_1 \\
  f_2(x) & \text{if } x \in A_2.
\end{cases}
\]

The function \( h \) is well-defined since the sets \( A_1 \) and \( A_2 \) do not have any common elements. It is easy to see that \( h \) is bijective. First, show that \( h \) is injective. Suppose \( h(a) = h(b) \) for some \( a, b \in A_1 \cup A_2 \). There are three cases:

- Case 1: both \( a \) and \( b \) are in \( A_1 \). Then \( h(a) = f_1(a) \) and \( h(b) = f_1(b) \) by definition of \( h \). Then, since \( f_1 \) is injective, we get \( a = b \).
• Case 2: both $a$ and $b$ are in $A_2$. This case is similar: since $f_2$ is injective, we get $a = b$.

• Case 3: one of the elements is in $A_1$, and the other – in $A_2$. We denote the one that is in $A_1$ by $a$. So, we have $a \in A_1$ and $b \in A_2$. Then $h(a) = f_1(a) \in B_1$, and $h(b) = f_2(b) \in B_2$, but since $B_1 \cap B_2 = \emptyset$, in this case the equality $h(a) = h(b)$ is impossible.

Now let us prove that $h$ is surjective. Let $b \in B_1 \cup B_2$. Then either $b \in B_1$ or $b \in B_2$. If $b \in B_1$, since $f_1$ is surjective, there exists $a \in A_1$ such that $f_1(a) = b$. Since $a \in A_1$, by definition of $h$, we have $h(a) = f_1(a) = b$. So, we proved that there exists $a \in A_1 \cup A_2$ such that $h(a) = b$. The case $b \in B_2$ is similar (replace $A_1$ with $A_2$ everywhere).

5. Section 13.2, Problem 4

**Solution:** Proof by contradiction. Suppose the set $\mathbb{I}$ of all irrational numbers was countable. We have: $\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$; we know that $\mathbb{Q}$ is countable, and the union of two (or in fact, finitely many, or even countably many) countable sets is countable. Then $\mathbb{R}$ would be countable, but that contradicts Cantor’s Theorem that $\mathbb{R}$ is uncountable.

6. Section 13.2, Problem 8

**Solution:** The statement is true. We know (proved in Section 13.2 and in class) that $\mathbb{Q}$ is countably infinite, and so is $\mathbb{Z}$. Then the set $\mathbb{Z} \times \mathbb{Q}$ is countably infinite by Theorem 13.5.

7. Section 13.2, Problem 12

**Solution:** There are many ways to solve this problem. Here is one of them.

We know that there exists a bijective function $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ (see the proof of Theorem 13.5). For each $k \in \mathbb{N}$, let $B_k = \{(k, m) \mid m \in \mathbb{N}\} \subset \mathbb{N} \times \mathbb{N}$, and let $A_k = f^{-1}(B_k)$. That is, $A_k$ is the inverse image of the set of elements of $\mathbb{N} \times \mathbb{N}$ that have the first coordinate $k$ (the same $k$ that is used to label our set). For example, $A_1$ is the inverse image under $f$ of the set of pairs of the form $(1, m)$ with $m \in \mathbb{N}$. Then we claim that $\mathbb{N} = \bigcup_{k=1}^{\infty} A_k$ is a partition that satisfies the requirements.

We need to check two things: that it is indeed a partition, and that each $A_k$ is countably infinite. (It is already automatic that we have $\aleph_0$ sets – they are indexed by natural numbers). First, let us check that $A_k$ is countably infinite for each $k \in \mathbb{N}$. This is easy: it is a pre-image of a countably infinite set $\{(k, m) \mid m \in \mathbb{N}\}$ under a bijective function, hence it has the same cardinality, i.e. is countably infinite.
Now let us check that the sets $A_k$ form a partition of $\mathbb{N}$. This means, we need to check two statements: 1) the union of these sets is all of $\mathbb{N}$, and 2) for $k \neq j$, $A_j \cap A_k = \emptyset$.

To prove these statements first note that the sets $B_k$ form a partition of $\mathbb{N} \times \mathbb{N}$ (make sure you can prove this – check the above two conditions.)

Now both of our statements are true simply by definition of a bijective function: the sets $A_k$ cannot have any common points simply because the sets $B_k$ do not have any common points, and $f$ is a function. The union of $A_k$ is all of $\mathbb{N}$ because the domain of $f$ is $\mathbb{N}$, and every point in $\mathbb{N} \times \mathbb{N}$ lies in the image of $f$ because $f$ is bijective, and in particular, surjective. Thus for each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $f(n) \in B_k$. Then $n \in A_k$ by definition of the inverse image.

8. Section 13.3, Problem 4

**Solution:** The statement is true. Proof: since $B$ contains an infinite set $A$, $B$ is infinite. On the other hand, since $B$ is contained in a countable set $C$, $B$ is countable (was proved in class, and also see the "criterion for countability" (Notes 2) on the common website).

9. Section 13.3, Problem 8

**Solution:** The statement is False.

We first prove that the set of *finite* sequences of integers is countable. Proof of this fact: Let $A_n$ denote the set of finite sequences of length $n$. Thus, $A_1$ is the set of sequences consisting of a single number, and $A_{10}$ is the set of sequences of ten integers. Then $A_n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ (the product of $n$ copies of $\mathbb{Z}$), by definition of Cartesian product of $n$ sets. We know (Corollary 13.1) that each such product is countably infinite. The set of all finite sequences is the union $\bigcup_{n=1}^{\infty} A_n$ of these sets, and by Problem 12 (see below), this set is countable.

Now we are ready to prove that the set of all infinite sequences of integers is uncountable. Assume to the contrary that it was countable. Then the set of all sequences of integers would have been countable as a union of two countable sets: the set of infinite sequences and the set of finite sequences, which we just proved is countable. But we know from Cantor’s diagonal argument that the set of all sequences of integers is uncountable – a contradiction!

Two notes: 1. the "contradiction" part of this argument is very similar to Problem 1 where we used the same contradiction argument to prove that the set of all irrational numbers is uncountable. 2. We used Cantor’s diagonal argument to prove that the set of sequences of *digits* is uncountable. There are two ways to use it to prove that the set of all sequences of *integers* is uncountable: one way is just to modify the argument itself to deal with sequences of integers. Another way is to say that the set of sequences of digits is contained in the set of sequences of integers; thus the
set of sequences of integers contains an uncountable set, and is therefore uncountable (make sure you know why! See the previous problem.)

10. Let $A$ be a non-empty set. Prove that $|A| \leq |A \times A|$. 

**Solution:**

- It suffices to find an injection from $A$ to $A \times A$.
- Define $f : A \to A \times A$ by $f(a) = (a, a)$.
- Let $a, b \in A$ and assume $f(a) = f(b)$. Thus $(a, a) = (b, b)$ and so we must have $a = b$. Hence $f$ is injective.

11. Let $A, B$ be sets. Prove that

if $|A - B| = |B - A|$ then $|A| = |B|$. 

Hint: draw a careful picture.

**Solution:** Given a bijection $f : (A - B) \to (B - A)$ define

$$g : A \to B \quad g(x) = \begin{cases} f(x) & x \in (A - B) \\ x & x \notin (A - B) \end{cases}$$

- Let $g : A \to B$ be defined as above. We need to show that $g$ is injective and surjective.
- Injective. Let $x, z \in A$ and assume $g(x) = g(z)$. This image must be in $B$, but it may either be in $A$ or not in $A$ (that is, either $y \in A \cap B$ or $y \in B - A$).
  - Assume $g(x) = g(z) \notin A$. Then both $x, z \in A - B$ (otherwise their images under $g$ would be in $A$). Hence $g(x) = f(x)$ and $g(z) = f(z)$. Since $f$ is injective, it follows that $x = z$. 

Now assume that \( g(x) = g(z) \in A \). Then both \( x, z \in A \) (otherwise their images under \( g \) would be in \( B - A \)). Then \( g(x) = x \) and \( g(z) = z \) and so \( x = z \).

Hence \( g \) is injective.

- **Surjective.** Let \( y \in B \). Either \( y \in A \) or \( y \not\in A \) (that is, either \( y \in A \cap B \) or \( y \in B - A \)).
  - Assume \( y \in A \) then let \( x = y \). By the definition of \( g \), \( g(x) = x = y \).
  - Now assume \( y \not\in A \), then since \( f \) is surjective, there exists \( x \in A - B \) so that \( f(x) = y \). Now since \( x \in A - B \), it follows that \( g(x) = f(x) = y \).

Hence \( g \) is surjective.

- Hence \( g(x) \) is bijective as required.

12. Prove that if \( A_n \) is countable for all \( n \in \mathbb{N} \), then \( A = \bigcup_{n=1}^{\infty} A_n \) is also countable. You may assume each \( A_n \) is non-empty (or just leave it out).

  **Hint.** Try to arrange the elements of \( A \) in a table, then use it to define a function \( f : \mathbb{N} \times \mathbb{N} \to A \), and show this function is surjective. Why does the result follow?

  **Solution:** If \( A \) is finite we are done so assume \( A \) is infinite. We may list the elements of \( A_n = \{a_{n,1}, a_{n,2}, a_{n,3}, \ldots \} \) in an infinite list –if \( A_n \) is finite just keep repeating the last element. Therefore

  \[
  A = \bigcup_{n=1}^{\infty} A_n = \{a_{n,k} : n, k \in \mathbb{N}\}.
  \]

Now define \( f : \mathbb{N} \times \mathbb{N} \to A \) by \( f(n, k) = a_{n,k} \). Clearly \( f \) is onto. We know \( \mathbb{N} \times \mathbb{N} \) is countable and so there is a bijection \( g : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \). Therefore \( f \circ g : \mathbb{N} \to A \) is onto and so \( A \) is countably infinite (recall \( A \) is infinite) by the Theorem proved in class (see Notes 2 on the common website).

13. Let \( B \) be a countably infinite set, and let \( f : A \to B \) be a surjective function such that \( f^{-1}(\{x\}) \) is a countable set for every \( x \in B \). Prove that \( A \) is countable.

  **Hint.** Use the previous problem.

  **Solution:** Let \( B = \{b_n : n \in \mathbb{N}\} \) where the \( b_n \)’s are all distinct (recall that it is given that \( B \) is countably infinite, so we can list its elements). Since \( f \) is surjective, \( A = \bigcup_{n=1}^{\infty} f^{-1}(\{b_n\}) \). Each \( f^{-1}(\{b_n\}) \) is countable by hypothesis. So we may apply the previous problem to conclude that \( A \) is countable.