Math 220 Final Study Guide


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1 Topics .......................... 1

1.1 Topics

• Chapter 1
  – Section 1.1: Sets, elements, infinite sets, finite sets, when two sets are equal, natural numbers, integers, real numbers, rational numbers, cardinality of a set, empty set, set builder notation (reading and writing it), interval notation, and all notation for sets.
  – Section 1.2: Ordered pair, cartesian product, notation for cartesian product, cardinality of a cartesian product, ordered triple, \((A \times B) \times C \neq A \times (B \times C)\).
  – Section 1.3: Subset, proper subset, notation for subset, empty set is subset of all sets, number of subsets of a finite set.
  – Section 1.4: Power set, notation for the power set, cardinality of a power set.
  – Section 1.5: Union of sets, intersection of sets, set difference, notation for all these.
  – Section 1.6: Complement of a set, universal set, notation for the complement.
  – Section 1.7: Venn diagrams, how to draw venn diagrams.
  – 1.8: Indexed sets, union of indexed sets (definition and how to calculate), intersection of indexed sets (definition and how to calculate)

• Chapter 2
  – Section 2.1: What is a statement, open sentence.
  – Section 2.2: Truth values of statements combined with and, or, not.
  – Section 2.3: Conditional statement, hypothesis, conclusion, truth values of a conditional statement (implication), different phrasings of a conditional statement as english sentences and determining the hypothesis and conclusion. (any odd number for practice)
  – Section 2.4: Biconditional statements, truth values of biconditional statements, different phrasings of biconditional statements as english sentences.
Section 2.5: Truth tables.
Section 2.6: Logical equivalence, DeMorgan’s laws, contrapositive law, commutative laws, distributive laws, associative laws.
Section 2.7: Universal quantifier, existential quantifier, the effect of the order of quantifiers, determining whether a statement with quantifiers is true or false, reading quantifiers from an English sentence.
Section 2.8: A statement in the form $P(x) \Rightarrow Q(x)$ is understood as $\forall x, P(x) \Rightarrow Q(x)$
Section 2.9: Expressing a statement written in mathematical logic as an English sentence and vice versa. (5,9,12)
Section 2.10: Negating a statement written as an English sentence, negating a statement written with symbolic logic (including quantifiers and implications).
Section 2.11: Logical inference like if $P \Rightarrow Q$ is true and $P$ is true then what can we conclude about $Q$? ($Q$ is true). See Section for more examples.

- Chapter 4: Direct proof, even/odd integers, same parity, opposite parity, divides, divisor, multiple, prime, composite, greatest common divisor, least common multiple, the division algorithm, how to prove a statement directly, clearly state all assumptions in the proof, how to use cases in a proof.

- Chapter 5: Contrapositive, how to write the contrapositive, how to prove a statement using the contrapositive, integer congruence modulo $n$ ($a \equiv b \pmod{n}$), how integer congruence modulo $n$ relates to the remainder in the division algorithm when you divide by $n$, Section 5.3 is about mathematical writing (while you should try to apply all of these to your homework item 4 would be nice to see on exams), knowing when to stop trying to prove a statement directly and switch to trying to prove a statement with the contrapositive.

- Chapter 6: Contradiction, how to prove a statement using contradiction, what is a contradiction, rational number, irrational number, when to try contradiction instead of a direct proof or contrapositive proof.

- Chapter 7: Proving biconditional statements (if and only if), proving equivalent statements (the following are equivalent, see page 123 for an example), proving existence statements, proving uniqueness statements, constructive proofs, non-constructive proofs.

- Chapter 8: Proving $a \in A$, proving $A \subseteq B$, proving $A = B$, and perfect numbers.

- Chapter 9: Determining whether a statement is true or false, what is a disproof, how to disprove a statement $P$, how to disprove a universal statement or conditional statement ($\forall x, P(x)$ or $P(x) \Rightarrow Q(x)$) with a counterexample, disproving an existence statement ($\exists x, P(x)$), and disproving a statement using contradiction.

- Chapter 10: Weak induction (standard induction), strong induction, proof by smallest counterexample, factorial, binomial coefficients $\binom{n}{k}$, fundamental theorem of arithmetic, Fibonacci numbers, and other number sequences defined recursively.

- Chapter 11
Section 11.0: What is a relation $R$ of a set $A$ ($R \subseteq A \times A$).

Section 11.1: Properties of relations which include reflexive, symmetric, and transitive.

Section 11.2: Equivalence relations, an equivalence class containing $a$ written $[a]$, and finding all equivalence classes.

Section 11.3: The fact that two equivalence classes are equal $[a] = [b]$ if and only if $a$ is related to $b$, partitions, and the fact that the collection of equivalence classes for a relation on set $A$ is a partition of $A$.

Section 11.4: The integers modulo $n$, the set $\mathbb{Z}_n$, and addition/multiplication for elements in $\mathbb{Z}_n$.

Chapter 12

Section 12.1: Functions, domain, codomain, range, the definition of a function as a relation from $A$ to $B$ ($f \subseteq A \times B$), and when two functions are considered equal.

Section 12.2: Definition of injective, definition of surjective, and definition of bijective.

Section 12.3: Pigeonhole principle.

Section 12.4: Composition of functions, the identity function $i_A$, and the fact that if $f$ and $g$ are injective (surjective) then $f \circ g$ is injective (surjective).

Section 12.5: A function is bijective if and only if its inverse relation is a function (i.e. the inverse exists).

Section 12.6: Image and preimage.

Chapter 13: Definition of $|A| = |B|$ in terms of a bijective function, how to prove $|A| = |B|$, how to prove $|\mathbb{Z}| = |\mathbb{N}|$, $|\mathbb{R}| = |(0, 1)|$, countably infinite, uncountable sets, if $A$ and $B$ are countably infinite sets then so is $A \times B$ and $A \cup B$, how to prove $|A| < |B|$, and the fact that $|A|$ is less than the cardinality of its power set.

2 How to study for a proof based exam

The following two steps are the best way to prepare for a proof based exam.

1. Know all the definitions and notation.
   - Generate an example which satisfies the definition.
   - If possible, generate an example which doesn’t satisfy the definition.

2. Practice
   - When you practice focus on understanding proofs, so then you may apply similar techniques on a different problem. Straight memorization of proofs from class or on the homework won’t be as effective.
   - Where to find practice problems
     - On our website find links to collections of old final exam and old final exams (Old final exams used a different book so you may find slightly different material and terminology).
– You can redo old homework question, then check your homework solutions or the online solutions. Often you can gain a deeper understanding of a problem the second time or make more sense of it.
– You can do some of the practice problems in the next section.

3 Practice Problems

Too many? Start with the underlined problems and homework problems. After (and after reviewing the definitions) focus on the section you feel less sure on.

Chapter 7 and later is new.

• Chapter 1
  – Section 1.2: 5, 7
  – Section 1.3: 9, 11
  – Section 1.4: 11, 17, 19
  – Section 1.5: 3
  – Section 1.7: 9, 11, draw a diagram when \( A \cap B = \emptyset \) and \( B \subset C \).
  – 1.8: 7, 9, 13

• Chapter 2
  – Section 2.3: For more practice with different phrasings of a conditional statement as english sentences and determining the hypothesis and conclusion. (any odd number for practice)
  – Section 2.5: 11
  – Section 2.6: 11, 13
  – Section 2.7: 5, 7, 9
  – Section 2.9: 5, 9, 12
  – Section 2.10: 9, 11

• Chapter 4: 19, 20, 27

• Chapter 5: 17, 19, 23, 25

• Chapter 6: 5, 9, 11, 17, 19, 21

• Chapter 7: 11, 14, 28, 33

• Chapter 8: 1, 5, 17, 23, 24, 27

• Chapter 9: 1, 5, 9, 15, 29, 33

• Chapter 10: 3, 11, 19, 37. Prove that if \( a_1 = 1 \), \( a_2 = 5 \), and \( a_n = 5a_{n-1} - 6a_{n-2} \) for \( n > 2 \) then \( a_n = 3^n - 2^n \).
4 Common Misconceptions and Mistakes

Items 6 and later are new.

1. There is a difference between absolute value and cardinality though they both use the same sign.
   - $|\{-5\}| = 1$ because $\{-5\}$ is a set with one element.
   - $|-5| = 5$ because $-5$ is a number.

2. There is a difference between the symbols $\in$ and $\subseteq$. Check out page 13 example 1.3 in the book for more examples.
   - $\{1\} \subseteq \{1, 2, 3\}$ and $1 \in \{1, 2, 3\}$ are both correct.
   - $1 \subseteq \{1, 2, 3\}$ is incorrect.

3. An implication, its negation, and its converse are all logically different. Though, an implication and its contrapositive are logically equivalent.

4. The negation of “For every positive $\epsilon$ we have $|\epsilon + 2| > 2$” is “There exists a positive $\epsilon$ where $|\epsilon + 2| \leq 2$”. The negation is not “There exists an $\epsilon \leq 0$, $|\epsilon + 2| \leq 2$”.

5. The negation of $(P \Rightarrow Q)$ is $(P$ and $\sim Q)$. The negation is not $(Q \Rightarrow P)$ which is the converse nor is it $(\sim P \Rightarrow \sim Q)$. The truth table will show the difference.
6. Assuming or using the conclusion to prove a statement. Often times this is not intentional and rewriting an argument can fix this issue.

**Example 1:** Show that if \( x = 2 \) then \( x^2 + 5 = 3 + 3x \).

*Incorrect proof:* (Assumes the conclusion) Let \( x = 2 \) then

\[
\begin{align*}
  x^2 + 5 &= 3 + 3x \\
  (2)^2 + 5 &= 3 + 3(2) \\
  9 &= 9.
\end{align*}
\]

*Correct proof:* Let \( x = 2 \) then

\[
\begin{align*}
  x^2 + 5 &= (2)^2 + 5 \\
  &= 9 \\
  &= 3 + 3(2) \\
  &= 3 + 3x.
\end{align*}
\]

**Example 2:** If \( n \) is odd then \( 8 | (n^2 - 1) \).

*Incorrect proof:* (Assumes the conclusion) Let \( n \) be odd so \( n = 2k + 1 \) for \( k \in \mathbb{Z} \). By the definition of divides \( 8 | (n^2 - 1) \) means \( n^2 - 1 = 8m \) for some \( m \in \mathbb{Z} \). Then

\[
\begin{align*}
  n^2 - 1 &= 8m \\
  (2k + 1)^2 - 1 &= 8m \\
  4k^2 + 4k &= 8m \\
  k^2 + k &= 2m.
\end{align*}
\]

So \( k^2 + k \) is even.

Say that \( k \) is even then \( k^2 \) is even. An even plus an even is even so \( k^2 + k \) is even.

Say that \( k \) is odd then \( k^2 \) is odd. An odd plus an odd is even so \( k^2 + k \) is even.

Since \( k^2 + k \) is even we have \( 8 | (n^2 - 1) \).

*Correct proof:* Let \( n \) be odd so \( n = 2k + 1 \) for \( k \in \mathbb{Z} \).

\[
\begin{align*}
  n^2 - 1 &= (2k + 1)^2 - 1 \\
  &= 4k^2 + 4k \\
  &= 4(k^2 + k).
\end{align*}
\]

Say that \( k \) is even then \( k^2 \) is even. An even plus an even is even so \( k^2 + k \) is even.

Say that \( k \) is odd then \( k^2 \) is odd. An odd plus an odd is even so \( k^2 + k \) is even.

Thus in all cases \( k^2 + k \) is even so \( k^2 + k = 2m \) for some \( m \in \mathbb{Z} \).

Then \( n^2 - 1 = 4(k^2 + k) = 8m \). By the definition of divides \( 8 | (n^2 - 1) \).
7. Forgetting to prove both directions of a biconditional statement. Remember that one way to show $A = B$ is to show the two directions $A \subseteq B$ and $B \subseteq A$.

8. For a function $f : A \to B$ showing that for $a \in A$ we can find some $f(a) \in B$ is not showing sujectivity. That is always true since $f$ is a function, and is not the definition for surjectivity. To show surjectivity we show that for any $b \in B$ that there is some element $a \in A$ which maps to $b$, $f(a) = b$, that every element in the codomain is mapped to by some element in the domain (the range equals to codomain).

9. The scratch work needed to prove a function is surjective is different from how to prove a function is surjective. For example we want to show $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{1\}$ where $f(x) = \frac{x - 1}{x - 2}$ is surjective. Scratch work should not be included in a proof.

**Scratch work**

\[
\begin{align*}
y &= \frac{x - 1}{x - 2} \\
x &= \frac{2y - 1}{y - 1} \\
y(x - 2) &= x - 1 \\
x - 2 &= x - 1 \\
x(y - 1) &= 2y - 1 \\
x &= \frac{2y - 1}{y - 1}
\end{align*}
\]

**Proof.** Let $y \in \mathbb{R} - \{1\}$ and consider $x = \frac{2y - 1}{y - 1}$. First we will show that $x \in \mathbb{R} - \{2\}$ (We have to show this because it is a needed fact that isn’t super obvious) by contradiction. Assume that instead $x = 2$ so

\[
\begin{align*}
2 &= \frac{2y - 1}{y - 1} \\
2y - 2 &= 2y - 1 \\
1 &= 2
\end{align*}
\]

which is impossible. Hence $x = \frac{2y - 1}{y - 1}$ is in the domain. Now we just have to show $f(x) = y$. We have

\[
\begin{align*}
f(x) &= f\left(\frac{2y - 1}{y - 1}\right) \\
&= \left(\frac{2y - 1}{y - 1}\right) - 1 \\
&= \left(\frac{2y - 1}{y - 1}\right) - 2 \\
&= \frac{2y - 1 - (y - 1)}{y - 1} \\
&= \frac{2y - 1 - 2(y - 1)}{y - 1} \\
&= \frac{y}{y - 1} \\
&= y.
\end{align*}
\]

Hence, $f$ is surjective. \(\square\)
10. Counterexamples only work for disproving universal statements like \( \forall x, P(x) \) or conditional statements \( P(x) \Rightarrow Q(x) \). Counterexamples are not able to disprove all types of statements.

11. An example is not a proof. The exception is when proving existence statements like \( \exists x, P(x) \) which asks for an example of \( x \) where \( P(x) \) is true.

5  Not sure if you are ready?

- As you practice, keep a list of problems that you stumble on so you can try them again later. If this list gets shorter you are on the right track.

- If you are able so solve the problem or produce a proof for problems in a timely manner without the aid of solutions, then you should feel more ready.

- Afraid to get stuck on an exam problem? Make a list of proof strategies you have used. Run through this mental list on the exam. Here are some off the top of my head. (use cases, even or odd, use cases determined remainders when you divide by \( n \), contradiction, contrapositive, direct proof, induction, is it a prove/disprove problem?, pigeonhole principle)