Homework 9

1. (Chapter 10: Question 8) If \( n \in \mathbb{N} \), then
\[
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.
\]

Proof. We will prove this by using induction on \( n \).

**Base step:** When \( n = 1 \) the left hand side is \( \frac{1}{2!} = \frac{1}{2} \). When \( n = 1 \) the right hand side is \( 1 - \frac{1}{(1+1)!} = \frac{1}{2} \) which proves the equality is true when \( n = 1 \).

**Inductive step:** Let \( n \geq 1 \). We will assume that
\[
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.
\]
Using our inductive assumption we have
\[
\left( \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} \right) + \frac{n+1}{((n+1)+1)!} = 1 - \frac{1}{((n+1)+1)!}.
\]
By induction we have thus shown that \( \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!} \) for all natural numbers \( n \).

\[\square\]

2. (Chapter 10: Question 10) For any integer \( n \geq 0 \), it follows that \( 3 \mid (5^{2n} - 1) \).

Proof. We will assume to the contrary that there is some integer \( n \geq 0 \) where 3 does not divide \( 5^{2n} - 1 \). This means that there is some smallest integer \( k \geq 0 \) such that \( 3 \nmid (5^{2k} - 1) \). Since \( k \) is smallest for any integer \( i \) with \( 0 \leq i < k \) we know \( 3 \mid (5^{2i} - 1) \).

Note that \( k \neq 0 \) since if \( k = 0 \) we have \( 5^{2k} - 1 = 0 \) and \( 3 \mid 0 \). Since \( k \neq 0 \) we can assume that \( k > 0 \). Since \( k > 0 \) we know that \( k - 1 \geq 0 \) and \( k - 1 \) is less than \( k \) so by our previous assumptions \( 3 \mid (5^{2(k-1)} - 1) \).

By definition of divides we have for some integer \( m \) that
\[
5^{2(k-1)} - 1 = 3m
\]
\[
25(5^{2(k-1)} - 1) = 25(3m)
\]
\[
5^{2k} - 25 = 75m
\]
\[
5^{2k} - 1 = 75m + 24
\]
\[
= 3(25m + 8)
\]
which contradicts our assumption that \( 3 \nmid (5^{2k} - 1) \). Hence \( 3 \mid (5^{2n} - 1) \) for all integers \( n \geq 0 \).

\[\square\]

3. (Chapter 10: Question 18) Suppose \( A_1, A_2, \ldots, A_n \) are sets in some universal set \( U \), and \( n \geq 2 \). Prove that
\[
A_1 \cup A_2 \cup \cdots \cup A_n = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.
\]
Proof. We will prove this by using induction on \( n \).

**Base step:** \( n = 2 \). We know whenever we have two sets \( A_1 \) and \( A_2 \) in the universal set \( U \) by DeMorgan’s law \( A_1 \cup A_2 = A_1 \cap A_2 \).

**Inductive step:** Let \( n \geq 2 \). We will assume that give any \( n \) subsets \( B_1, B_2, \ldots, B_n \) of \( U \) that \( B_1 \cup B_2 \cup \cdots \cup B_n = B_1 \cap B_2 \cap \cdots \cap B_n \). Our goal is to show that given any \( n+1 \) subsets \( A_1, A_2, \ldots, A_{n+1} \) of \( U \) that \( A_1 \cup A_2 \cup \cdots \cup A_{n+1} = A_1 \cap A_2 \cap \cdots \cap A_{n+1} \). So let \( A_1, A_2, \ldots, A_{n+1} \) be any \( n+1 \) sets which are subsets of the universal set \( U \). Note that then \( A_1, A_2, \ldots, A_n \) is a collection of \( n \) subsets of \( U \) so we can apply our inductive assumption to this collection of sets. Using DeMorgan’s law and our inductive assumption we have that

\[
A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1} = (A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1}
\]

\[
= (A_1 \cup A_2 \cup \cdots \cup A_n) \cap A_{n+1}
\]

\[
= A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}
\]

which proves our statement by induction. \( \square \)

4. (Chapter 10: Question 22) If \( n \in \mathbb{N} \), then

\[
(1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8})(1 - \frac{1}{16}) \cdots (1 - \frac{1}{2^n}) \geq \frac{1}{4} + \frac{1}{2^{n+1}}.
\]

Proof. We will prove this by using induction on \( n \).

**Base step:** When \( n = 1 \) the left hand side is \((1 - \frac{1}{2}) = \frac{1}{2}\). When \( n = 1 \) the right hand side is \(\frac{1}{4} + \frac{1}{2^{1+1}} = \frac{1}{2}\) and since \(\frac{1}{2} \geq \frac{1}{2}\) the inequality is true for \( n = 1 \).

**Inductive step:** Let \( n \geq 1 \). We will assume that \((1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8})(1 - \frac{1}{16}) \cdots (1 - \frac{1}{2^n}) \geq \frac{1}{4} + \frac{1}{2^{n+1}}\). Our goal is to show that \((1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8})(1 - \frac{1}{16}) \cdots (1 - \frac{1}{2^n}) \cdots (1 - \frac{1}{2^{n+1}}) \geq \frac{1}{4} + \frac{1}{2^{n+1}}\). Using our inductive assumption we have

\[
\left[(1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8})(1 - \frac{1}{16}) \cdots (1 - \frac{1}{2^n})\right] \left(1 - \frac{1}{2^{n+1}}\right) \geq \left[\frac{1}{4} + \frac{1}{2^{n+1}}\right] \left(1 - \frac{1}{2^{n+1}}\right)
\]

\[
= \frac{1}{4} + \frac{1}{2^{n+1}} - \frac{1}{4} \frac{1}{2^{n+1}} - \frac{1}{2^{n+1}} \frac{1}{2^{n+1}}
\]

\[
= \frac{1}{4} + \frac{1}{2^{n+1}} \left(1 - \frac{1}{4} - \frac{1}{2^{n+1}}\right)
\]

\[
= \frac{1}{4} + \frac{1}{2^{n+1}} \left(\frac{3}{4} - \frac{1}{2^{n+1}}\right).
\]

Because \( n \geq 1 \) we know \(\frac{1}{2^{n+1}} \leq \frac{1}{2^{n+1}} = \frac{1}{4}\) so

\[
\frac{1}{4} + \frac{1}{2^{n+1}} \left(\frac{3}{4} - \frac{1}{2^{n+1}}\right) \geq \frac{1}{4} + \frac{1}{2^{n+1}} \left(\frac{3}{4} - \frac{1}{4}\right)
\]

\[
= \frac{1}{4} + \frac{1}{2^{n+1}} \left(\frac{1}{2}\right)
\]

\[
= \frac{1}{4} + \frac{1}{2^{(n+1)+1}}
\]

Thus we have shown \((1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8})(1 - \frac{1}{16}) \cdots (1 - \frac{1}{2^n}) \cdots (1 - \frac{1}{2^{n+1}}) \geq \frac{1}{4} + \frac{1}{2^{(n+1)+1}}\). Hence by induction \((1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8})(1 - \frac{1}{16}) \cdots (1 - \frac{1}{2^n}) \cdots (1 - \frac{1}{2^{n+1}}) \geq \frac{1}{4} + \frac{1}{2^{n+1}}\) for all \( n \in \mathbb{N} \). \( \square \)

5. Prove that if \( m \in \mathbb{N} \) is a multiple of \( 4 \) then \( F_m \) is a multiple of \( 3 \).

The **Fibonacci numbers** are defined to be \( F_1 = 1 \), \( F_2 = 1 \), and \( F_n = F_{n-1} + F_{n-2} \) for \( n > 2 \).
6. Show using induction for any integer \( n \geq 0 \) we have that \( F_m = F_{4k} \) is a multiple of 3. We will prove this by inducting on \( k \).

**Base step:** When \( k = 1 \) we have \( m = 4k = 4 \) and \( F_m = F_4 = F_3 + F_2 = (F_2 + F_1) + F_2 = (1 + 1) + 1 = 3 \) which is a multiple of 3.

**Inductive step:** Let \( k \geq 1 \) we will assume that \( F_{4k} \) is a multiple of 3 and show that \( F_{4(k+1)} \) is also a multiple of 3. Since \( F_{4k} \) is a multiple of 3 we know \( F_{4k} = 3x \) for some integer \( x \). Using the Fibonacci recurrence we have

\[
F_{4(k+1)} = F_{4k+4} = F_{4k+3} + F_{4k+2} = (F_{4k+2} + F_{4k+1}) + (F_{4k+1} + F_{4k}) = ((F_{4k+1} + F_{4k}) + F_{4k+1}) + (F_{4k+1} + F_{4k}) = 3F_{4k+1} + 2F_{4k} = 3F_{4k+1} + 2(3x) = 3(F_{4k+1} + 2x).
\]

Hence \( F_{4(k+1)} \) is also a multiple of 3. Therefore, by induction we can conclude that \( F_{4k} \) is a multiple of 3 for all \( k \in \mathbb{N} \).

\[ \square \]

6. Show using induction for any integer \( n \geq 0 \) and \( x, y \in \mathbb{R} \) that

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\]

We define for integers \( n \) and \( k \) the binomial coefficient to be

\[
\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & n \geq 0 \text{ and } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}
\]

where \( n \text{ factorial} \) is \( n! = n(n-1)(n-2) \ldots (2)(1) \) with \( 0! = 1 \).

**Proof.** Let \( x \) and \( y \) be real numbers.

**Base step:** When \( n = 0 \) the left hand side is \((x+y)^0 = 1\). The right hand side is \( \sum_{k=0}^{0} \binom{0}{k} x^k y^{0-k} = \binom{0}{0} x^0 y^0 = 1 \), so we can see the equality is true when \( n = 0 \).

**Inductive step:** Let \( n \geq 0 \). We will assume that \( (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \). Our goal is to show that \( (x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k} \).

First note that by the given definition of binomial coefficients that \( 1 = \binom{n}{0} = \binom{n+1}{0} = \binom{n}{n} = \binom{n+1}{n+1} \). Also by the definition we have that

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} \left( \frac{1}{n-k+1} + \frac{1}{k} \right) = \frac{n!}{(k-1)!(n-k)!} \left( \frac{n+1}{k(n-k+1)} \right) = \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}.
\]
Using our inductive assumption and the facts we established above we have

\[(x+y)^{n+1} = (x+y)^n (x+y)\]
\[= \left( \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \right) (x+y)\]
\[= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}\]
\[= \sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}\]
\[= \left( \sum_{k=1}^{n} \binom{n}{k-1} \right) x^{n+1-k} + \left( \sum_{k=0}^{n} \binom{n}{k} \right) x^k y^{n+1-k} + x^{n+1}\]
\[= \left( \sum_{k=1}^{n+1} \binom{n+1}{k} \right) x^{n+1-k} + x^{n+1}\]
\[= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}.\]

Hence, by induction \((x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\) for all integers \(n \geq 0\).

7. The Tribonacci numbers are defined as \(T_1 = 1, T_2 = 1, T_3 = 1,\) and \(T_n = T_{n-1} + T_{n-2} + T_{n-3}\) for \(n > 3\). Show that for all \(n \in \mathbb{N}\) that \(T_n \leq 2^n\).

**Proof.** We will prove the statement by inducting on \(n\).

**Base steps:** When \(n = 1\) we have \(T_1 = 1\) which is less than \(2^1 = 2\). When \(n = 2\) we have \(T_2 = 1\) and \(1 \leq 2^2\). When \(n = 3\) we have \(T_3 = 1\) and \(1 \leq 2^3\). Hence, the statement is true for \(n\) equal to 1, 2, and 3.

**Inductive step:** Let \(n \geq 3\). We will assume for all positive integers \(k \leq n\) that \(T_k \leq 2^k\). Our goal is to show that \(T_{n+1} \leq 2^{n+1}\). Using our inductive hypotheses and the definition of Tribonacci numbers we have

\[T_{n+1} = T_n + T_{n-1} + T_{n-2}\]
\[\leq 2^n + 2^{n-1} + 2^{n-2}\]
\[= 2^{n-2}(4 + 2 + 1)\]
\[= 2^{n-2}(7)\]
\[\leq 2^{n-2}(8)\]
\[= 2^{n+1}.\]

Hence, \(T_{n+1} \leq 2^{n+1}\). This proves by induction that \(T_n \leq 2^n\) for all positive integers \(n\).
8. For \( n \in \mathbb{N} \) let \( B_n \) be the set of all \( n \)-digit binary numbers. For example \( B_3 = \{000, 001, 010, 100, 011, 101, 110, 111\} \).

(a) Show that \( |B_n| = 2^n \).

**Proof.** We will prove this by induction on \( n \). First a small notational note. If we have two binary words \( x \) and \( y \) we will let \( xy \) denote their concatenation. Meaning if \( x = 0100 \) and \( y = 111 \) then \( xy = 0100111 \). Further this will make \( 1x = 10100 \) and \( 0x = 00100 \).

**Base steps:** When \( n = 1 \) the set \( B_1 = \{0, 1\} \) has cardinality \( 2 = 2^1 \). Thus, the statement is true for \( n = 1 \).

**Inductive step:** Let \( n \geq 1 \). We will assume that \( |B_n| = 2^n \) and will show that \( |B_{n+1}| = 2^{n+1} \). Let

\[
S_0 = \{ x \in B_{n+1} : x \text{ starts with a } 0 \} = \{0x : x \in B_n\}
\]

and

\[
S_1 = \{ x \in B_{n+1} : x \text{ starts with a } 1 \} = \{1x : x \in B_n\}.
\]

Any word in the set \( B_{n+1} \) will start with a 1 or start with a 0 so \( S_0 \cup S_1 = B_{n+1} \). Since a word cannot both start with a 0 and a 1 we know \( S_0 \cap S_1 = \emptyset \).

We can see that \( S_0 = \{0x : x \in B_n\} \) has the same number of elements as \( B_n \) so \( |S_0| = |B_n| = 2^n \).

Similarly, \( S_1 = \{1x : x \in B_n\} \) has the same number of elements as \( B_n \) so \( |S_1| = |B_n| = 2^n \).

**Fact:** If a finite set \( C \) can be written as a union \( C = A \cup B \) with \( A \cap B = \emptyset \) then \( |C| = |A| + |B| \).

Since \( S_0 \cup S_1 = B_{n+1} \) and \( S_0 \cap S_1 = \emptyset \) we know

\[
|B_{n+1}| = |S_0| + |S_1| = 2^n + 2^n = 2^{n+1}.
\]

Hence, by induction \( |B_n| = 2^n \) for all natural numbers \( n \). \( \square \)

(b) (Chapter 10: Question 32) Prove that the number of \( n \)-digit binary numbers that have no consecutive 1’s is the Fibonacci number \( F_{n+2} \).

**Proof.** We will use the set \( B_n \) and the notation defined for concatenation of binary words introduced in part (a). We will let

\[
C_n = \{ x \in B_n : x \text{ has no consecutive 1’s} \}.
\]

Our goal is to prove that \( |C_n| = F_{n+2} \) which we will prove by using induction on \( n \).

**Base steps:** When \( n = 1 \) the set \( C_1 = \{0, 1\} \) has cardinality \( 2 = F_3 = F_{1+2} \). When \( n = 2 \) the set \( C_2 = \{00, 01, 10\} \) has cardinality \( 3 = F_4 = F_{2+2} \). Thus, the statement is true for \( n = 1 \) and \( n = 2 \).

**Inductive step:** Let \( n \geq 2 \). We will assume for all positive integers \( k \leq n \) that \( |C_k| = F_{k+2} \) and will show that \( |C_{n+1}| = F_{(n+1)+2} = F_{n+3} \). Let

\[
R_0 = \{ x \in C_{n+1} : x \text{ starts with a } 0 \}.
\]

We will show that \( R_0 = \{0x : x \in C_n\} \). Let \( y \in R_0 \) then certainly \( y = 0x \) for some binary word \( x \) of length \( n \). Since \( y \in C_{n+1} \) we know \( y \) does not have any consecutive 1’s so \( x \) doesn’t have any consecutive 1’s so \( x \in C_n \). Thus, \( y \in \{0x : x \in C_n\} \) and \( R_0 \subseteq \{0x : x \in C_n\} \). Let \( y \in \{0x : x \in C_n\} \) then \( y = 0x \) for some \( x \in C_n \). Since \( x \) has no consecutive 1’s and putting a 0 at the front of \( x \) doesn’t create any consecutive 1’s we know \( y \) doesn’t have any consecutive 1’s, starts with a 0, and has length \( n+1 \). Hence, \( y \in R_0 \) and \( R_0 = \{0x : x \in C_n\} \). We can see now that \( |R_0| = |C_n| = F_{n+2} \) by our inductive assumption.

Let

\[
R_1 = \{ x \in C_{n+1} : x \text{ starts with a } 1 \}.
\]
We will show that $R_1 = \{10x : x \in C_{n-1}\}$. Let $y \in R_1$ then certainly $y = 1x$ for some binary word $x$ of length $n$. Since $y \in C_{n+1}$ we know $y$ does not have any consecutive 1’s meaning that the second number in the word $y$ must be a 0 else we have consecutive 1’s. Now we have $y = 10z$ for some $z \in C_{n-1}$ which implies that $y \in \{10x : x \in C_{n-1}\}$ and $R_1 \subseteq \{10x : x \in C_{n-1}\}$. Say instead that $y \in \{10x : x \in C_{n-1}\}$. Then $y = 10x$ for some $x \in C_{n-1}$. Since $x$ has no consecutive 1’s and placing a 10 before $x$ doesn’t create any consecutive 1’s we know $y = 10x$ has no consecutive 1’s, $y$ starts with a 1, and has length $n+1$. Hence $y \in R_1$ and $R_1 = \{10x : x \in C_{n-1}\}$. We can see now that $|R_1| = |C_{n-1}| = F_{(n-1)+2} = F_{n+1}$ by our inductive assumption.

Any word in the set $C_{n+1}$ will start with a 1 or start with a 0 so $R_0 \cup R_1 = C_{n+1}$. Since a word cannot both start with a 0 and a 1 we know $R_0 \cap R_1 = \emptyset$.

**Fact:** If a finite set $C$ can be written as a union $C = A \cup B$ with $A \cap B = \emptyset$ then $|C| = |A| + |B|$. Since $R_0 \cup R_1 = C_{n+1}$ and $R_0 \cap R_1 = \emptyset$ we know

$$|C_{n+1}| = |R_0| + |R_1| = F_{n+2} + F_{n+1} = F_{n+3}.$$

Hence, by induction $|C_n| = F_{n+2}$ for all natural numbers $n$. □

(c) Use parts (a) and (b) to show $F_{n+2} \leq 2^n$ where $F_n$ means the $n$th Fibonacci number.

**Proof.** We will use the same notation for sets introduced in parts (a) and (b). In part (a) we had $B_n$ the collection of all binary words length $n$ and $|B_n| = 2^n$. In part (b) we had $C_n$ the collection of all binary words length $n$ with no consecutive 1’s and $|C_n| = F_{n+2}$. Since any binary word of length $n$ with no consecutive 1’s is a binary word length $n$ we have $C_n \subseteq B_n$ for all natural numbers $n$. Since these are finite sets their cardinalities satisfy the inequality $|C_n| \leq |B_n|$ which implies that $F_{n+2} \leq 2^n$ for all natural numbers $n$. □