Homework 10
Solutions

1. Section 11.1, #12
Prove that the relation “divides” on the set \( \mathbb{Z} \) is reflexive and transitive.
Solution:
For any \( x \in \mathbb{Z} \), \( x = 1x \), so \( x|x \), hence the relation is reflexive.
Suppose \( x, y, z \in \mathbb{Z} \) are such that \( x|y \) and \( y|z \). Then there exist integers \( a \) and \( b \) such that \( xa = y \) and \( yb = z \). Then \( ab \in \mathbb{Z} \) and \( x(ab) = z \), so \( x|z \). We conclude the relation is transitive.

2. Suppose \( R \) is a symmetric and transitive relation on a set \( A \), and for every \( x \in A \) there exists some \( a \in A \) such that \( aRx \). Prove that \( R \) is reflexive.
Solution:
Let \( x \in A \). By the information given in the question, there exists \( a \in A \) such that \( aRx \). Since \( R \) is symmetric, \( xRa \). Since \( R \) is transitive, and since \( xRa \) and \( aRx \), we conclude \( xRx \). Therefore, \( R \) is reflexive.

3. Let \( A = \{1, 2, 3\} \). Give an example of a relation on \( A \) that is symmetric and transitive, but not reflexive.
Solution:
One such relation is \( R = \emptyset \). That is, no element of \( A \) relates to any other.
Another example is \( R = \{(a, b) : \min\{a, b\} > 1\} = \{2, 3\} \times \{2, 3\} \).

4. Below, we describe four relations on the integers. For each, prove or disprove that it is an equivalence relation. For the equivalence relation(s), describe \([26]\), either by writing out all its terms, or by noticing that it is a familiar set.
   
   (a) \( Q \subseteq \mathbb{Z} \times \mathbb{Z} \), \( Q = \{(a, b) : \gcd(a, b) > 1\} \)
Solution:
\( Q \) is not an equivalence relation. In particular, it is not transitive. Note that \( 2, 5, 10 \in \mathbb{Z} \) and \( (2, 10) \in Q \) and \( (10, 5) \in Q \), but \( (2, 5) \notin Q \).

   (b) \( R \subseteq \mathbb{Z} \times \mathbb{Z} \), \( R = \{(a, b) : |a - b| < 2\} \)
Solution:
\( R \) is not an equivalence relation, because it is not transitive. Note \( 1, 2, 3 \in \mathbb{Z} \), and \( (1, 2) \in R \) and \( (2, 3) \in R \), but \( (1, 3) \notin R \).

   (c) \( S \subseteq \mathbb{Z} \times \mathbb{Z} \), \( S = \{(a, b) : a^2 = b^2\} \)
Solution:
\( S \) is an equivalence relation. For any \( a \in \mathbb{Z} \), \( a^2 = a^2 \), so \( S \) is reflexive. If \( a, b \in \mathbb{Z} \),...
and aSb, then \( a^2 = b^2 \), so \( b^2 = a^2 \), so \( bSa \). Therefore \( S \) is symmetric. Suppose there exist \( a, b, c \in \mathbb{Z} \) such that \( aSb \) and \( bSc \). Then \( a^2 = b^2 \) and \( b^2 = c^2 \), so \( a^2 = c^2 \), hence \( aSc \). So, \( S \) is symmetric. We conclude \( S \) is an equivalence relation.

We note that \([26] = \{a \in \mathbb{Z} : a^2 = 26^2\}\), so \([26] = \{-26, 26\}\).

(d) \( T \subseteq \mathbb{Z} \times \mathbb{Z}, T = \{(a, b) : a^2 \equiv b^2 \mod 4\}\)

Solution:

\( T \) is an equivalence relation. Let \( a \in \mathbb{Z} \). Since \( 4 \mid a^2 - a^2 \), we see that \( a^2 \equiv a^2 \mod 4 \), so \( aTa \), hence \( T \) is reflexive. Suppose \( a, b \in \mathbb{Z} \) and \( aTb \). That is, \( a^2 \equiv b^2 \mod 4 \). Then \( 4 \mid (a^2 - b^2) \), which means \( 4x = a^2 - b^2 \) for some integer \( x \). Then \( -x \) is also an integer, and \( 4(-x) = b^2 - a^2 \), so \( b^2 \equiv a^2 \mod 4 \), hence \( bTa \), so \( T \) is symmetric. Finally, suppose \( a, b, c \in \mathbb{Z} \) and \( aTb, bTc \). By our definition of \( T \), that means \( a^2 \equiv b^2 \mod 4 \) and \( b^2 \equiv c^2 \mod 4 \). By the definition of congruence, that means \( 4 \mid (a^2 - b^2) \) and \( 4 \mid (b^2 - c^2) \). So, \( 4 \mid [(a^2 - b^2) + (b^2 - c^2)] \). That is, \( 4 \mid a^2 - c^2 \), so \( a^2 \equiv c^2 \mod 4 \), hence \( aTc \) and \( T \) is transitive. We conclude \( T \) is an equivalence relation.

If \( a \) is even, then \( a^2 \equiv 0 \mod 4 \). If \( a \) is odd, then \( a^2 \equiv 1 \mod 4 \). So, \( T = \{(a, b) : a \equiv b \mod 2\}\). Therefore, \([26]\) is the set of all even integers.

5. Cigol is a student who dislikes truth tables. Cigol is considering statements that are made out of statements \( P \) and \( Q \) (possibly repeated), together with the symbols \( \lor, \land, \) and \( \sim \). Cigol will call two statements “related” if they agree in at least three of the four columns of a truth table. For example: the statements \( P \lor \sim Q \) and \((P \lor Q) \land \sim (P \land Q)\) agree in three cases (\( P \) and \( Q \) both false; \( P \) true and \( Q \) false; \( P \) false and \( Q \) true), so Cigol calls them related.

Show that Cigol’s relation is not an equivalence relation.

Solution:

Consider these three statements: \( S_1 = P \lor \sim P, \quad S_2 = P \lor Q, \quad S_3 = \sim (P \land Q) \). The truth table below shows that all \( S_2 \) relates to \( S_1 \), and \( S_1 \) relates to \( S_3 \), but \( S_2 \) does not relate to \( S_3 \).

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \lor (\sim P) )</th>
<th>( P \lor Q )</th>
<th>( \sim (P \land Q) )</th>
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6. Section 11.2, #4

Let \( A = \{a, b, c, d, e\} \). Suppose \( R \) is an equivalence relation on \( A \). Suppose also that \( aRd \) and \( bRc, eRa, \) and \( cRe \). How many equivalence classes does \( R \) have?

Solution:
Only one. Using symmetry and transitivity, we see that every element relates to every other.

7. List all the partitions of \( A = \{a, b, c\} \).

**Solution:**

- \( \{\{a\}, \{b\}, \{c\}\} \)
- \( \{\{a, b\}, \{c\}\} \)
- \( \{\{a\}, \{b, c\}\} \)
- \( \{\{a, c\}, \{b\}\} \)
- \( \{\{a, b, c\}\} \)

8. Let \( A = \{0, 1\}^3 \): that is, \( A \) is the set of all ordered triples with entries from 0 and 1. Then define a relation \( R \subseteq A \times A \) such that \( xRy \) if and only if \( x \) and \( y \) have the same number of 0s. Note that \( R \) is an equivalence relation.

Give the partition of \( A \) created by the equivalence classes of \( R \).

**Solution:**

\[
\{ \{(0,0,0)\}, \{(1,0,0),(0,1,0),(0,0,1)\}, \{(0,1,1),(1,0,1),(1,1,0)\}, \{(1,1,1)\} \}
\]

9. Section 11.4, #2 (see Page 192 for examples)

**Write the addition and multiplication tables for \( \mathbb{Z}_3 \).**

**Solution:**

\[
\begin{array}{c|ccc}
\hline
\end{array}
\]

10. Section 11.4, #6

**Suppose \([a], [b] \in \mathbb{Z}_6 \) and \([a] \cdot [b] = [0] \). Is it necessarily true that either \([a] = [0] \) or \([b] = [0] \)?**

**Solution:**
No: it could be (for instance) that $[a] = 2$ and $[b] = 3$.

Remark: when $n$ is a prime, if $[a] \cdot [b] = [0]$, then one of the two classes must be zero. This is not the case when $n$ is composite.