This Examination paper consists of 7 pages (including this one). Make sure you have all 7.

INSTRUCTIONS:

No memory aids allowed. No calculators allowed. No communication devices allowed. Please circle the name of your instructor below.

MARKING:

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Names of Instructors: Jim Bryan, Mark Maclean
Q1 [20 marks]

Consider the curve parameterized by the vector function

\[ \mathbf{r}(t) = t \cos(t) \mathbf{i} + t \sin(t) \mathbf{j} + t \mathbf{k}, \quad -4\pi < t < 4\pi \]

(a) (5 points). Find a surface on which the parameterized curve lives and sketch the curve and the surface in the space below.

\[ x^2 + y^2 = t^2 \cos^2 t + t^2 \sin^2 t \]

\[ = t^2 = z^2 \]

so curve lies on the cone \( x^2 + y^2 = z^2 \)

\[ \mathbf{r}'(0) = \langle 0, 2, 0 \rangle \]

(b) (2 points). Find \( |\mathbf{r}'(t)| \). You must simplify your answer to get credit for this problem. There should be no sines or cosines in your answer.

\[ \mathbf{r}'(t) = \langle \cos t - t \sin t, \sin t + t \cos t, 1 \rangle \]

\[ |\mathbf{r}'(t)| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 1} \]

\[ = \sqrt{\cos^2 t - 2t \cos t \sin t + t^2 \sin^2 t + \sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t + 1} \]

\[ = \sqrt{1 + t^2 + 1} \]

\[ |\mathbf{r}'(t)| = \sqrt{2 + t^2} \]
(f) (2 points). Find $\mathbf{T}(0)$.

\[
\mathbf{T}(0) = \frac{\mathbf{T}'(0)}{|\mathbf{T}'(0)|} = \frac{\langle 1, 0, 1 \rangle}{\sqrt{2}} = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle
\]

(g) (2 points). Deduce from the above answers what $\mathbf{N}(0)$ is (hint: use the decomposition of acceleration into normal and tangential components).

\[
\mathbf{T}''(0) = \frac{d^2 \mathbf{s}}{dt^2}(0) \mathbf{T} + k(0) \left| \mathbf{T}'(0) \right|^2 \mathbf{N}
\]

\[
\langle 0, 1, 0 \rangle = 0 \cdot \mathbf{T} + (1 \cdot 2) \mathbf{N}
\]

So \[\mathbf{N}(0) = \langle 0, 1, 0 \rangle\]

(h) (2 points). Find the equation of the osculating plane at $t = 0$.

It is the plane through the origin, perpendicular to

\[
\mathbf{T}(0) \times \mathbf{N}(0) = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle \times \langle 0, 1, 0 \rangle = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0
\end{vmatrix}
\]

\[
= \langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle
\]

\[-\frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} z = 0
\]

\[
x = z \quad \text{is the equation of the plane}
\]
Q2 [6 marks]

Suppose that \( \mathbf{r}(t) \) is a vector valued function with \( |\mathbf{r}(t)| = A \) and \( |\mathbf{r}'(t)| = B \) where \( A \) and \( B \) are constants.

Compute the quantity

\[
\mathbf{r}'(t) \cdot (\mathbf{r}(t) \times (\mathbf{r}'(t) \times \mathbf{r}'(t)))
\]

Your answer should be expressed solely in terms of \( A \) and \( B \) (hint: it helps to draw a picture of the vectors \( \mathbf{r}(t) \), \( \mathbf{r}'(t) \), and \( \mathbf{r}(t) \times \mathbf{r}'(t) \) for some fixed value of \( t \)).

\[
\text{Since } \mathbf{r} \cdot \mathbf{r} = A^2 \\
2 \mathbf{r}' \cdot \mathbf{r} = 0 \quad \text{so} \\
\mathbf{r}' \text{ is perpendicular} \\
to \frac{\mathbf{r}}{A} \\
\]

\[
\mathbf{r} \times (\mathbf{r} \times \mathbf{r}') \\
\text{has length } A^2 B \text{ and} \\
is in opposite direction of } \mathbf{r}' \quad \text{thus} \\
\mathbf{r} \times (\mathbf{r} \times \mathbf{r}') = -A^2 \mathbf{r}' \\
\text{thus} \quad \mathbf{r}' \cdot (\mathbf{r} \times (\mathbf{r} \times \mathbf{r}')) = -A^2 \mathbf{r}' \cdot \mathbf{r}' = \boxed{-A^2 B^2}
\]
Q3  [12 marks]

Let 

$$\vec{r}(t) = \langle 2t^2, (1-t)(1+t), -2t^2 + 1 \rangle, \quad t \geq 0$$

be a parameterized curve. Reparameterize the curve by arclength beginning at the point $(0,0,1)$ and going in the positive $t$ direction. Geometrically describe this curve.

$$\vec{r}'(t) = \langle 4t, -2t, -4t \rangle$$

$$|\vec{r}'(t)| = 2t \sqrt{4+9} = 6t$$

$$s(t) = \int_{u=0}^{t} 6u \, du = 3u^2 \bigg|_{0}^{t} = 3t^2$$

$$s = 3t^2, \quad t = \sqrt{\frac{s}{3}}$$

$$\vec{r}(s) = \left\langle \frac{2}{3} s, 1 - \frac{s}{3}, -\frac{2}{3} s + 1 \right\rangle$$

This is the line passing through $(0,1,1)$ in the direction $\langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \rangle$.

It is also given as the intersection of the planes $x + 2 = 1$, $x + 2y = 2$. 

Q4  [12 marks]

For each of the six plots below, clearly mark each plot with the letter corresponding to the vector-valued function that generates that plot. Note that not all letters will be used.

\[ \mathbf{A}: \mathbf{F}(t) = \langle t^2, t^3 \rangle, \quad -5 \leq t \leq 5 \]
\[ \mathbf{B}: \mathbf{F}(t) = \langle e^t, e^{-t} \rangle, \quad -5 \leq t \leq 5 \]
\[ \mathbf{C}: \mathbf{F}(t) = \langle e^t, \ln(t) \rangle, \quad t > 0 \]
\[ \mathbf{D}: \mathbf{F}(t) = \langle \ln(t), \sqrt{t} \rangle, \quad t > 0 \]
\[ \mathbf{E}: \mathbf{F}(t) = \langle 5 \sin(t), t^2 \rangle, \quad -\pi \leq t \leq \pi \]
\[ \mathbf{F}: \mathbf{F}(t) = \langle 1 + \sqrt{t}, t^2 - 4t \rangle, \quad t > 0 \]
\[ \mathbf{G}: \mathbf{F}(t) = \langle e^{-t} + t, e^t - t \rangle, \quad -5 \leq t \leq 5 \]
\[ \mathbf{H}: \mathbf{F}(t) = \langle 5 \cos(t), t^2 \rangle, \quad -\pi \leq t \leq \pi \]