Vector Valued Functions of One Variable.

We will denote a vector $\mathbf{V}$ in $\mathbb{R}^3$ as

$$\mathbf{V} = (V_1, V_2, V_3) = \langle V_1, V_2, V_3 \rangle = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$$

$V_1$ is the $x$ component of $\mathbf{V}$

$V_2$ is the $y$ component of $\mathbf{V}$

$V_3$ is the $z$ component of $\mathbf{V}$

We will use the same notation for a point in $\mathbb{R}^3$.

$$\mathbf{p} = (p_1, p_2, p_3).$$

The $\mathbf{V}$ has magnitude $||\mathbf{V}|| = \sqrt{V_1^2 + V_2^2 + V_3^2}$ and its direction is given by the straight line joining the origin $(0,0,0)$ to the point $(V_1, V_2, V_3)$, pointing outwards.

We will denote by $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$ as the scalar product / dot product / inner product of two vectors

$\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (V_1, V_2, V_3)$

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 V_1 + u_2 V_2 + u_3 V_3.$$

$$= ||\mathbf{u}|| \cdot ||\mathbf{v}|| \cdot \cos \theta$$

where $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. 
One has \( \| \vec{v} \|^2 = \langle \vec{v}, \vec{v} \rangle \).

One also has the vector product of two vectors \( \vec{u} \) and \( \vec{v} \):

\[
\vec{u} \times \vec{v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3 \\
\end{vmatrix} = (u_2v_3 - v_2u_3) \hat{i} \\
+ (v_1u_3 - u_1v_3) \hat{j} \\
+ (u_1v_2 - v_1u_2) \hat{k}
\]

\[
= \begin{pmatrix}
u_2v_3 - v_2u_3 \\
v_1u_3 - u_1v_3 \\
u_1v_2 - v_1u_2
\end{pmatrix}
\]

\( \vec{u} \times \vec{v} = -\vec{v} \times \vec{u} \) is a vector \( \perp \) to both \( \vec{v} \) and \( \vec{u} \) and has magnitude

\[
\| \vec{u} \| \| \vec{v} \| \sin \theta
\]

A vector valued function \( \vec{F}(t) \) of one variable \( t \) is written as

\[
\vec{F}(t) = (F_1(t), F_2(t), F_3(t))
\]

\( \vec{F}(t) \) is continuous iff each of its components \( F_1(t), F_2(t) \) and \( F_3(t) \) is continuous.

\( \vec{F}(t) \) is differentiable iff each of its components \( F_1(t), F_2(t) \) and \( F_3(t) \) is differentiable.
and one has
\[
\frac{d}{dt} \vec{F}(t) + \vec{F}'(t) = (F_1'(t), F_2'(t), F_3'(t))
\]

Then the usual rules for derivatives apply.

Consider two vector fields \( \vec{F}(t) \) and \( \vec{G}(t) \) of one variable \( t \).

One has

1) \[
\frac{d}{dt} (\vec{F}(t) + \vec{G}(t)) = \frac{d}{dt} \vec{F}(t) + \frac{d}{dt} \vec{G}(t)
\]

2) Product rules

i) \[
\frac{d}{dt} \langle \vec{F}(t), \vec{G}(t) \rangle = \langle \vec{F}'(t), \vec{G}(t) \rangle + \langle \vec{F}(t), \vec{G}'(t) \rangle
\]

ii) \[
\frac{d}{dt} (\vec{F}(t) \times \vec{G}(t)) = \vec{F}'(t) \times \vec{G}(t) + \vec{F}(t) \times \vec{G}'(t)
\]
1) **Parametrized Curves.**

**Def:** A parametrized curve \( \vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t)) \) in \( \mathbb{R}^3 \) is a differentiable function, from an open interval \((a, b) \subset \mathbb{R}\) into \( \mathbb{R}^3 \).

\[ \vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t)) \quad \text{if} \quad a < t < b \]

(or \( \vec{\gamma}(t) \))

The variable \( t \) is called the parameter of the curve.

One can visualize \( \vec{\gamma}(t) \) as describing the motion of a particle in space with respect to time \( t \).

**Def:** If \( \vec{\gamma}(t) \) is a parametrized curve, its first derivative \( \vec{\gamma}'(t) = (\gamma_1'(t), \gamma_2'(t), \gamma_3'(t)) \) is called the tangent vector (or the velocity vector) of the curve \( \vec{\gamma}(t) \) at the point \( \vec{\gamma}(t) \) at time \( t^* \).
Curve in the plane

\( \vec{y}(t) = (y_1(t), y_2(t), c) \) is a curve in the plane \( z = c \). If \( y_3(t) = 0 \), then \( \vec{y}(t) \) is a curve in the \( xy \) plane.

Similarly for other components.

Line

The line passing through the point \( \vec{p} = (p_1, p_2, p_3) \) and with direction \( \vec{v} \) is given by \( \vec{p} + t\vec{v}, t \in \mathbb{R} \)

\[ \vec{l}(t) = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3), \quad -\infty < t < \infty \]

If the interval of \( t \) is finite then one has a part of the line, called line segment.

One has \( \vec{l}(t) = \vec{v} \) for all time \( t \).

Circle

One can parametrize the circle in the \( xy \) plane:

\[ x^2 + y^2 = r^2 \]

as

\[ \vec{y}(t) = (r \cos t, r \sin t, 0), \quad 0 \leq t \leq 2\pi \]
The velocity at time \( t \) is

\[
\vec{y}'(t) = (-x \sin t, x \cos t, 0)
\]

# Note:
\( \vec{y}(0) = \vec{y}(2\pi) \)

**Def:** A curve \( \vec{y}(t) \) defined for \( a \leq t \leq b \) such that \( \vec{y}(a) = \vec{y}(b) \) is called a *closed curve*.

**Ex. 4:** Consider the curve \( \vec{y}(t) = (1, t, 4t^2) \), \( 0 \leq t \leq 1 \)
and the curve \( \vec{x}(t) = (1, t^2, 4t^4) \), \( 0 \leq t \leq 1 \).

Both the curves \( \vec{x}(t) \) and \( \vec{y}(t) \) trace out the same path in \( \mathbb{R}^3 \) in the same amount of time \( t \),

Since in both the cases the *amplitude* of the curve is:
\[ z = 4y^2, \quad x = 1 \]

But note:
\[
\vec{y}'(t) = (0, 1, 8t), \quad 0 \leq t \leq 1
\]
\[
\vec{x}'(t) = (0, 2t, 16t^3), \quad 0 \leq t \leq 1
\]

So the velocities are different.
Example 5: Consider the curves
\[
\vec{\gamma}(t) = (\cos t, \sin t, 0) \quad 0 \leq t \leq 2\pi
\]
\[
\vec{\beta}(s) = (\cos(2s), \sin(2s), 0) \quad 0 \leq s \leq \pi
\]
\[\vec{\gamma}'(t)\text{ and } \vec{\beta}'(s)\text{ trace out the same path/curve.}\]
\[x^2 + y^2 = 1\text{ in the } x, y\text{ plane.}\]
\[\vec{\gamma}'(t)\text{ & } \vec{\beta}'(s)\text{ are two different parametrizations of the unit circle.}\]

One has,
\[
\vec{\gamma}'(t) = (-\sin t, \cos t, 0) \quad 0 \leq t \leq 2\pi
\]
\[
\vec{\beta}'(s) = (-2\sin(2s), 2\cos(2s), 0) \quad 0 \leq s \leq \pi
\]
and
\[
\vec{\gamma}'(2s) = \vec{\beta}'(2s) \quad 0 \leq s \leq \pi
\]
\[
or, \quad \vec{\gamma}'(t) = \vec{\beta}'(t) \quad 0 \leq t \leq 2\pi
\]
The curve \(\vec{\beta}(s)\) moves twice as fast as \(\vec{\gamma}(t)\)
\[\text{i.e., } \vec{\beta}'(s) = 2 \vec{\gamma}'(2s)\]

The speed \(|\vec{\beta}'(s)| = 2\), \(|\vec{\gamma}'(t)| = 1\)
Similarly the curve $\gamma(t) = (-\sin t, \cos t, 0)$ for $0 \leq t \leq 2\pi$ is also a parametrization of the unit circle $x^2 + y^2 = 1$.

The starting pts $[\dot{\gamma}(0) = (1,0,0), \dot{\gamma}(0) = (0,1,0)]$ are different.

\[ \dot{\gamma}(t) = (-\cos t, -\sin t, 0) \]


# Parametrizations are not unique. One can have
We will describe a canonical (or but) parametrization later, in Arc-Length.

E.g. \( \vec{x}(t) = (t^3 - 4t, t^2 - 4) \), \( t \in \mathbb{R} \)

is a parametrized curve.

Note: \( \vec{x}(0) = 0, (0, 0) \).

So the curve intersects with itself.

At \((0, 0)\) \( \vec{x} \) has two tangent vectors.

So in general it doesn't make sense to talk about the tangent vector at a point on the curve, but rather tangent vector at time \( t \), the parameter.

Def: A curve \( \vec{y}(t) \) which do not intersect itself is called a simple curve.
Example: \( \mathbf{x}(t) = (t^3, t^2) \); \( t \in \mathbb{R} \) is a parametrized curve.

\[ \mathbf{x}'(0) = (0/0) \]; i.e., the velocity vector is 0 for \( t = 0 \).

**Def:** A parametrized curve \( \gamma(t) \) is called a **regular curve** if \( \mathbf{\gamma}'(t) \neq 0 \) for all \( t \).
2) **Arc Length**

If $\mathbf{v} = (v_1, v_2, v_3)$ is a vector in $\mathbb{R}^3$, its length is:

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

(distance from the origin $(0,0,0)$)

If $\mathbf{u}$ is another vector in $\mathbb{R}^3$, then the distance between $\mathbf{u}$ and $\mathbf{v}$ is: $||\mathbf{u} - \mathbf{v}||$

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**Example 1: Length of a Straight Line**

Consider the straight line $\ell(t) = \mathbf{p} + t\mathbf{v}$, $-\infty < t < +\infty$.

Let $\mathbf{p}$ be a point on the line, and $\mathbf{v}$ be the direction vector.

The line can be parameterized as $\ell(t) = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3)$.

Passing through $\mathbf{p}$, with direction $\mathbf{v}$.

The distance between two points $\ell(t_a)$ and $\ell(t_b)$ is:

$$\mathbf{a} = \ell(t_a) = (p_1 + ta v_1, p_2 + ta v_2, p_3 + ta v_3)$$

$$\mathbf{b} = \ell(t_b) = (p_2 + tb v_2, p_3 + tb v_3)$$

on the straight line is:

$$||\mathbf{a} - \mathbf{b}|| = ||\ell(t_a) - \ell(t_b)|| = ||t_a - t_b|| ||\mathbf{v}||$$

This is also called the **arc-length** of the straight line between time $t_a$ and time $t_b$. 
So the arc-length of the line between time \( t \) and \( t + \Delta t \) is

\[ l = \| \mathbf{v} \| \Delta t \]

\[ = \text{(length of tangent vector)} \times \text{(amount of time difference)} \]

The above example motivates the following definition of arc-length of a parametrized curve \( \mathbf{r}(t) \), \( a \leq t \leq b \)

To calculate the length of a curve \( \mathbf{r}(t) \), we divide the time interval into "small" segments, each of which corresponds to a small increment \( \Delta t := \frac{b-a}{n} \)

Length of the polygon inscribed in \( \mathbf{r}(t) \) with vertices \( \mathbf{r}(t_i) \), \( i = 0, 1, 2, \ldots, n \)

is then

\[ = \sum_{i=1}^{n} \left( \text{length of the vector } \mathbf{r}(t_i) - \mathbf{r}(t_{i-1}) \right) \times \Delta t \]
For $\Delta t$ small,
\[
\frac{\vec{y}(t_i) - \vec{y}(t_{i-1})}{\Delta t} = \frac{\vec{y}(t_{i+1}) - \vec{y}(t_{i-1})}{\Delta t}
\]
is "nearly" equal to the tangent vector $\vec{y}'(t)$.

So for $\Delta t$ or large number of divisions $n$.

Length of the polygon $\approx \sum_{i=1}^{n} \| \vec{y}'(t_i) \| \Delta t$

Letting $\Delta t \to 0$ we have

Length of the polygon $\to$ length of the curve $\vec{y}(t)$ between $t = a$ and $t = b$.

So we define

**Def**: The arc-length of a curve $\vec{y}(t)$ between $t = a$ and $t = b$ is

\[
\int_{a}^{b} \| \vec{y}'(t) \| \, dt
\]
Next, we introduce the arc-length parameter $s(t)$, which measures the distance along the curve from a fixed time, say $t_0$, to time $t$. This arc-length parameter gives a canonical (or best) parametrization of any regular curve, i.e., $\gamma'(t) \neq 0$ for any time $t = 0$.

**Def:** Arc-length parameter $s(t)$ of a regular parametrized curve $\gamma(t)$, $t_0 \leq t \leq t_1$ from to $u$

\[ s(t) = \int_{t_0}^{t} \| \gamma'(u) \| \, du \]

The arc-length parameter $s(t)$ is a differentiable function of $t$ and by the fundamental theorem of calculus

\[ s'(t) = \frac{ds}{dt} = \| \gamma'(t) \| = \text{speed of } \gamma(t) \]

To reparametrize a curve by arc-length.

Given a regular parametrized curve \( \vec{r}(t) \), \( a \leq t \leq b \),


**Step 1:** Find \( s(t) \), the arc-length parameter.

\[
s(t) = \int_{t_0}^{t} \| \vec{r}'(u) \| \, du \]


**Step 2:** We have \( s \) as a function of \( t \).

Solve for \( t \) and find \( t \) as a function of \( s \).

\( t(s) \)


**Step 3:** In the expression for the curve \( \vec{r}(t) \), replace \( t \) with \( t(s) \). Then \( \vec{r}(t) = \vec{r}(t(s)) \)

is a function of \( s \).

The curve \( \vec{r}(t(s)) \) is a curve parametrized by the arc length parameter \( s \).
Reparameterize the curve \( \vec{r}(t) = (3 \sin t, 4t, 3 \cos t) \) \( \forall \), 
\(-\infty < t < +\infty \), in terms of the arc-length parameter, measured from \( t = 0 \).

**Sol:**

\[
\vec{r}(t) = (3 \sin t, 4t, 3 \cos t) \\
\vec{r}'(t) = (3 \cos t, 4, -3 \sin t)
\]

**Speed:**

\[
\| \vec{r}'(t) \| = \sqrt{3^2 \cos^2 t + 4^2 + (-3)^2 \sin^2 t} \\
= \sqrt{9 \cos^2 t + 9 \sin^2 t + 16} = \sqrt{9 + 16} = \sqrt{25} \\
= 5.
\]

Thus, the arc-length parameter \( s(t) \) is given by

\[
s(t) = \int_0^t \| \vec{r}'(u) \| \, du = \int_0^t 5 \, du = 5t \\
= 5t.
\]

So \( t = s/5 \).

Hence \( \vec{r}(s) = (3 \sin (s/5), 4s/5, 3 \cos (s/5)) \)

is a reparameterization of \( \vec{r}(t) \) by the arc-length parameter \( s \).
8. Find a parametrization of the curve
$$\{ z = \sqrt{x^2+y^2} \} \cap \{ z = 1+y \}$$

cone \(\cap\) plane

**Sol:** What is the candidate for the parameter \( t \)?

We let \( x = t \). We need to find \( y \) and \( z \) in terms of \( t \).

Using the eqns:

\[
\sqrt{x^2+y^2} = z = 1+y
\]

\[
\Rightarrow \quad \sqrt{t^2+y^2} = 1+y
\]

\[
\Rightarrow \quad t^2+y^2 = (1+y)^2 = 1+2y+y^2
\]

\[
\Rightarrow \quad t^2 = 1+2y
\]

So,

\[
y = \frac{t^2-1}{2}
\]

Then

\[
z = 1+y = 1 + \frac{t^2-1}{2} = \frac{t^2+1}{2}
\]

Therefore a parametrization of the given curve is

\[
\gamma(t) = \left( t, \frac{1}{2}(t^2-1), \frac{1}{2}(t^2+1) \right) ; \quad -\alpha < t < +\alpha
\]
Find a parametrization of the line tangent to the helix \( \vec{r}(t) = (\cos t, \sin t, t) \) at the point \((1,0,0)\).

The point \((1,0,0)\) corresponds to the value of \( \vec{r}(t) \) at \( t=0 \), i.e., \( \vec{r}(0) = (1,0,0) \).

We have \( \vec{r}'(t) = (-\sin t, \cos t, 1) \)

so \( \vec{r}'(0) = (0,1,1) \).

So then we want to parametrize the line passing through the point \((1,0,0)\) and in the direction of the vector \( \vec{r}'(0) = (0,1,1) \), \(-\infty < t < +\infty\).

\[ l'(t) = (1,0,0) + t(0,1,1), \quad -\infty < t < +\infty \]

\[ = (1, t, t). \]
Suppose \( \vec{F}(t) \) is a vector valued function such that \( ||\vec{F}(t)|| = 1 \).

Then show that \( \vec{F}'(t) \perp \vec{F}(t) \) i.e.
their dot product / scalar product \( \langle \vec{F}'(t), \vec{F}(t) \rangle = 0 \)

**Sol.**

\[
||\vec{F}(t)|| = 1 \\
\Rightarrow ||\vec{F}'(t)||^2 = 1 \\
\Rightarrow \langle \vec{F}'(t), \vec{F}(t) \rangle = 1
\]

Differentiating both the sides we obtain:

\[
\frac{d}{dt} \left( \langle \vec{F}'(t), \vec{F}(t) \rangle \right) = 0
\]

\[
\Rightarrow \langle \vec{F}''(t), \vec{F}(t) \rangle + \langle \vec{F}'(t), \vec{F}'(t) \rangle = 0
\]

\[
\Rightarrow 2 \langle \vec{F}'(t), \vec{F}(t) \rangle = 0
\]

\[
\Rightarrow \langle \vec{F}'(t), \vec{F}(t) \rangle = 0
\]

hence \( \vec{F}'(t) \perp \vec{F}(t) \)

**Recall:**

- For any vector \( \vec{v} \)
  \[ \langle \vec{v}, \vec{v} \rangle = ||\vec{v}||^2 \]

- For any two vector fields \( \vec{F}(t) \) and \( \vec{G}(t) \)
  \[
  \frac{d}{dt} \langle \vec{F}(t), \vec{G}(t) \rangle \\
  = \langle \vec{F}'(t), \vec{G}(t) \rangle \\
  + \langle \vec{F}(t), \vec{G}'(t) \rangle
  \]
Geometrically, at any point $p^3 = (p_1, p_2, p_3)$ on the sphere, the tangent vector is $\perp$ to $p^3$.

**E.g. 11** Calculate: $$\frac{d}{dt} \langle \vec{n}(t), \vec{n}'(t) \times \vec{n}''(t) \rangle$$

**Sol.** One has

$$\frac{d}{dt} \langle \vec{n}(t), \vec{n}'(t) \times \vec{n}''(t) \rangle = \langle \vec{n}(t), \vec{n}'(t) \times \vec{n}''(t) \rangle + \langle \vec{n}'(t), \vec{n}'(t) \times \vec{n}''(t) \rangle$$

by product rule.

$$= 0 + \langle \vec{n}'(t), \vec{n}'(t) \times \vec{n}''(t) \rangle$$

$$= \langle \vec{n}(t), \vec{n}''(t) \times \vec{n}'''(t) \rangle$$

$$= \langle \vec{n}'(t), \vec{n}''(t) \times \vec{n}'''(t) \rangle$$
Example 12 Find the length of one revolution of the helix
\[ \vec{r}(t) = (a \cos t, a \sin t, \frac{bt}{2\pi}), \quad 0 \leq t \leq 2\pi \]

Here \( a, b \) are positive constants.

Solution: One has \( \vec{r}'(t) = (-a \sin t, a \cos t, \frac{b}{2\pi}) \)

Therefore the speed \( ||\vec{r}'(t)|| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + \frac{b^2}{(2\pi)^2}} \)

\[ = \sqrt{a^2 + \frac{b^2}{(2\pi)^2}} \]

The length of one revolution of the helix is given by

\[ \int_{0}^{2\pi} ||\vec{r}'(t)|| \, dt = \int_{0}^{2\pi} \sqrt{a^2 + \frac{b^2}{(2\pi)^2}} \, dt \]

\[ = 2\pi \sqrt{\frac{a^2 + b^2}{(2\pi)^2}} = \sqrt{(2\pi a)^2 + b^2} \]
when $b \to 0$ the helix becomes a circle of radius $a$, then
\[ \text{length} = 2\pi a = \text{circumference}. \]

when $a \to 0$, the helix becomes a line segment, then
\[ \text{length} = b = \text{height of the helix}. \]

\[ \text{Eq. 13} \quad \text{Given the curve } \mathbf{r}(t) = \left( t^2, \frac{4\sqrt{2}}{3} t^{3/2}, 2t \right), \quad t \geq 0 \]

reparametrize by the arc-length parameter from
\[ t = t_0 = 0. \]

\text{Sol:} \quad \text{The arc length parameter } s(t) \text{ is a function of time } t \text{ and is given by:}
\[ s(t) = \int_{0}^{t} \| \mathbf{r}'(u) \| \, du \]

one has \[ \mathbf{r}'(t) = \left( 2t, 2\sqrt{2} \sqrt{t}, 2 \right) \]

therefore,
\[ \| \mathbf{r}'(t) \| = \sqrt{(2t)^2 + (2\sqrt{2} \sqrt{t})^2 + 2^2} = \sqrt{4t^2 + 8t + 4} \]
\[ = 2\sqrt{t^2 + 2t + 1} = 2\sqrt{(t+1)^2} = 2(t+1) \]
Then,

$$\lambda(t) = \int_0^t \|\gamma'(u)\| \, du = \int_0^t 2 \,(u+1) \, du$$

$$= \, t^2 + 2t$$

i.e.

$$\lambda = t^2 + 2t$$

so,

$$\lambda + 1 = \, t^2 + 2t + 1 = (t+1)^2$$

hence,

$$t+1 = \sqrt{\lambda + 1}$$

$$\Rightarrow \quad t = \sqrt{\lambda + 1} - 1$$

Substituting \( t \) with \((\sqrt{\lambda + 1} - 1)\) gives us the reparametrization of the given curve by the parameter \( s \).

So,

$$\gamma(s) = \left( (\sqrt{\lambda + 1} - 1)^2, \, \frac{4\sqrt{2}}{3} \,(\sqrt{\lambda + 1} - 1)^{3/2}, \, 2(\sqrt{\lambda + 1} - 1) \right)$$

\( \square \).
If a parametrized curve \( \vec{r}(t) \) has unit speed, i.e.

\[
\| \vec{r}'(t) \| = 1 \quad \text{for all time } t,
\]

then the parameter of curve is essentially the arc length parameter.

On the other hand, one can also the following:

**Fact:** Suppose \( \vec{r}(s) \) is a parametrized curve, parametrized by arc length \( s \). Then \( \vec{r}(s) \) is a unit speed curve, i.e.

\[
\| \vec{r}'(s) \| = 1 \quad \text{for all values of } s.
\]

Now one has

\[
\lambda = \lambda(t) \quad \text{and} \quad \frac{ds(t)}{dt} = \| \vec{r}'(t) \| > 0
\]

Then \( t \) can be written as a function of \( s \), i.e. \( t = t(s) \)

and so,

\[
\frac{dt(s)}{ds} = \frac{1}{\frac{ds(t)}{dt}} = \frac{1}{\| \vec{r}'(t) \|}
\]

Therefore, using the chain rule we get

\[
\frac{d\vec{r}(s)}{ds} = \frac{d\vec{r}(s(t))}{dt} \frac{dt(s)}{ds} = \frac{d\vec{r}(s(t))}{dt} \frac{dt(s)}{ds}
\]
Hence
\[ \left\| \frac{d\vec{r}(s)}{ds} \right\| = \left\| \frac{d\vec{r}(t)}{dt} \cdot \frac{dt(s)}{ds} \right\| \]

\[ = \left\| \frac{d\vec{r}(t)}{dt} \right\| \left\| \frac{dt(s)}{ds} \right\| \]

\[ = \left\| \vec{r}'(t) \right\| \left\| \frac{1}{\| \vec{r}'(t) \|} \right\| \]

\[ = 1 \]

\[ \therefore \left\| \vec{r}(s) \right\| = 1 \text{ for any value of } s. \]
Fact:
A parametrized curve \( \vec{\gamma}(r), h_0 < r < h_1 \) is a reparametrization of a curve \( \vec{\gamma}(t), a < t < b \), if there exist functions \( f(t) \) and \( g(r) \) such that
\[
\begin{align*}
   r &= f(t) \\
   t &= g(r)
\end{align*}
\]
for all \( t \) and \( r \).

\[\begin{cases}
   \text{and} & g(f(t)) = t \\
   & f(g(r)) = r
\end{cases}\]

\[\begin{cases}
   \text{Both } f(t) \text{ and } g(r) \text{ are differentiable.}
\end{cases}\]

ii) \[\begin{cases}
   g(r) = f^{-1}(r) \\
   f(t) = g^{-1}(t)
\end{cases}\]

\[\begin{cases}
   \vec{\gamma}(r) = \vec{\gamma}(g(r)) \\
   \vec{\gamma}(t) = \vec{\gamma}(f(t))
\end{cases}\]

for all \( t \) and \( r \).
Two curves that are reparametrizations of each other have the same image, so they should have the same geometric properties.