\( \overrightarrow{F} \) is conservative

\[ \oint_C \overrightarrow{F} \cdot d\overrightarrow{r} = 0 \]
for any closed curve \( C \)

\( \int_{C} \overrightarrow{F} \cdot d\overrightarrow{r} \)
depends only on end pts of \( C \)
A continuous vector field $\mathbf{F} = (P, Q)$ defined in a domain/region $D$ is \textbf{Conservative} if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{in} \quad D$$

and the domain/region $D$ is \textbf{simply connected} (no holes).

\begin{itemize}
\item Simply connected
\item Not simply connected
\end{itemize}

**Def:** A domain $D$ in the plane is \textbf{simply connected} if the points inside of any closed loop in $D$, is in $D$, that is $D$ has no holes. Then closed loop in $D$ can be shrunk to a point without leaving $D$. 
If we remove a point from the plane, then it is no more simply connected.

Multiple holes
Not simply connected

Not simply connected since the domain has two pieces.
e.g. Consider the vector field \( \vec{F} = (x e^y, y e^x) \) defined in the plane. Is \( \vec{F} \) conservative?

Sol:

\[ \vec{F} = (x e^y, y e^x) \]

\[ P = x e^y \]

\[ Q = y e^x \]

Then \( \frac{\partial P}{\partial y} = x e^y \)

and \( \frac{\partial Q}{\partial x} = y e^x \)

So \( \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \) and therefore \( \vec{F} \) cannot be a conservative vector field.

e.g. Consider the vector field \( \vec{F} \) defined in the plane as

\[ \vec{F}(x, y) = (y e^x + \sin y, e^x + x \cos y + e^y) \]

Is \( \vec{F} \) conservative?
Sol:
\[ \frac{\partial P}{\partial y} = e^x + \cos y \]
\[ \frac{\partial Q}{\partial x} = e^x + \cos y \]

So, \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \). Since \( \vec{F} \) is a continuous vector field defined in the plane which is simply connected, therefore \( \vec{F} \) is conservative.

Hence there exists a differentiable function with continuous derivatives such that \( \vec{F} = \nabla f \) in the plane.

Then such a \( f \) must satisfy
\[ \frac{\partial f}{\partial x} = P \quad \text{and} \quad \frac{\partial f}{\partial y} = Q \]

Therefore,
\[ \frac{\partial f}{\partial x} = ye^x + \sin y \quad \text{and} \quad \frac{\partial f}{\partial y} = e^x + x \cos y + ey \]
Integrating \( \frac{\partial f}{\partial y} - y = 0 \) gives:

\[
f(x,y) = \int (ye^x + \cosh y) \, dy + g(y)
\]

\( \Rightarrow \) \( f(x,y) = ye^x + x \sinh y + g(y) \)

\( \Rightarrow \) \( \frac{df}{dy} = e^x + x \cosh y + g'(y) = e^x + x \cosh y + ey \)

\( \Rightarrow \) \( g'(y) = ey \)

\( \Rightarrow \) \( g(y) = e^y + C \)

Therefore, \( f(x,y) = e^x + x \cosh y + ey + C. \)

**Example:** Consider the vector field

\[
\vec{F}(x,y) = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)
\]

defined for all \( (x,y) \neq (0,0). \)

**Solution:** The vector field is defined and continuous everywhere except the origin \((0,0).\)
One has
\[
\frac{\partial P}{\partial y} = \frac{- (x^2 + y^2) - 2y (-y)}{(x^2 + y^2)^2} = \frac{-x^2 - y^2 + 2y^2}{(x^2 + y^2)^2}
\]
\[= \frac{y^2 - x^2}{(x^2 + y^2)^2}\]

Similarly,
\[
\frac{\partial P}{\partial x} = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}
\]

So,
\[
\frac{\partial P}{\partial y} = \frac{\partial P}{\partial x} \quad \text{for all } (x, y) \neq (0, 0)
\]

The domain has a hole at the origin, and so is not simply connected. Therefore, one cannot conclude anything. There one needs to check if really
\[
\int_C \mathbf{F} \cdot d\mathbf{r}
\]
for all closed loops $C$, for all closed loops $C$. 
For our loop, first trace the unit circle $y = R\cos t$ starting at $(0, R)$.

Let $C$ be: \( \vec{y}(t) = (R\cos t, R\sin t), \quad 0 \leq t \leq 2\pi \).

Then \( \vec{y}'(t) = (-R\sin t, R\cos t) \).

So,
\[
\int_{C} \vec{F} \cdot d\vec{r} = \int_{0}^{2\pi} \left( \frac{-R\sin t}{R^2}, \frac{R\cos t}{R^2} \right) \cdot (-R\sin t, R\cos t) \, dt
\]
\[
= \int_{0}^{2\pi} (\sin^2 t + \cos^2 t) \, dt = \int_{0}^{2\pi} 1 \, dt = 2\pi
\]

\[\downarrow\]

Very strange, a constant number.

So \( \vec{F} \) is not conservative as we have found one closed curve $C$ such that \( \int_{C} \vec{F} \cdot d\vec{r} \neq 0 \).
If we consider \( \vec{F} \) on the plane \( \mathbb{R}^2 - \{ \text{positive x-axis} \} \), then \( \vec{F} \) is conservative.

Since the domain \( D \) in this case is simply connected and
\[
\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},
\]

So there is some function defined on \( D \) (but not on \( \mathbb{R}^2 - \{0,0\} \)) such that \( \nabla \cdot \vec{F} = 0 \).

\[
\frac{\partial f}{\partial y} = Q = \frac{x}{x^2 + y^2} \quad \Rightarrow \quad f(x, y) = \int \frac{x}{x^2 + y^2} \, dy + g(x)
\]

\[
\Rightarrow \quad f(x, y) = \arctan \left( \frac{y}{x} \right) + g(x)
\]

\[
\Rightarrow \quad \frac{\partial f}{\partial x} = \frac{-y}{x^2 + y^2} + g'(x) = \frac{-y}{x^2 + y^2}
\]

\[
\Rightarrow \quad g'(x) = 0 \quad \Rightarrow \quad g(x) = C
\]
So, \( f(x,y) = \arctan \left( \frac{y}{x} \right) + C \)

\( \theta \) from polar coordinates

\[ (x,y) \]

So, \( \int_C F \cdot d\mathbf{s} \) makes more sense now

If \( C \) goes about once around the origin, then

\[ \int_C F \cdot d\mathbf{s} = 2\pi \]

we say in this case, the winding number is 1.

In general, for any curve \( C \)

\[ \text{Calculate} \rightarrow \int_C F \cdot d\mathbf{s} = 2\pi \cdot \{ \text{winding number of } C \text{ around the origin} \} \]

\[ \text{Geometry/Topology of } D \]

\[ \text{Winding } = 0 \]
\[ \text{Winding } = 1 \]
\[ \text{Winding } = 2 \]
\[ \text{Winding } = -1 \]
Show that
\[ \oint_C F \cdot ds = \int_C \frac{x}{1+x^2+y^2} \, dx + \left( \frac{y}{1+x^2+y^2} + \frac{y}{1+y^2} \right) \, dy \]
depends only on the end points of \( C \), for any given curve \( C \).

Then evaluate \( \oint_{C_1} F \cdot ds \) where \( C_1 \) is a curve from \((0,0)\) to \((3,4)\).

\[ F = (P, Q) \text{ is defined everywhere in the plane} \]

Here \( P = \frac{x}{1+x^2+y^2} \) and \( Q = Y = \frac{y}{1+x^2+y^2} - \frac{y}{1+y^2} \)

Then \( \frac{\partial P}{\partial y} = -\frac{2xy}{(1+x^2+y^2)^2} \)

and \( \frac{\partial Q}{\partial x} = -\frac{2xy}{(1+x^2+y^2)^2} \)
So \( P_y = Q_x \) and domain is the plane, which is simply connected.

Therefore \( \mathbf{F} \) is conservative and \( \int_C \mathbf{F} \cdot d\mathbf{r} \) depends only on the end pts of the curve \( C \).

Next, we need to find the potential \( f \), that is \( f \) satisfying:

\[
\nabla f = \mathbf{F}
\]

One then has \( \frac{\partial f}{\partial x} = P = \frac{x}{1+x^2+y^2} \)

\[
\Rightarrow f(x, y) = \int \frac{x}{1+x^2+y^2} \, dx + g(y)
\]

\[
= \frac{1}{2} \ln \left(1+x^2+y^2\right) + g(y)
\]

Then, \( \frac{\partial f}{\partial y} = \frac{y}{1+x^2+y^2} + g'(y) = Q \)

\[
\Rightarrow \frac{y}{1+x^2+y^2} + g'(y) = \frac{y}{1+x^2+y^2} - \frac{y}{1+y^2}
\]
\[ g'(y) = \frac{-y}{1+y^2} \]

\[ \Rightarrow g(y) = -\int \frac{y}{1+y^2} \, dy + c = -\frac{1}{2} \ln(1+y^2) + C \]

So,

\[ f(x,y) = \frac{1}{2} \ln(1+x^2+y^2) - \frac{1}{2} \ln(1+y^2) + C \]

\[ = \ln \sqrt{\frac{1+x^2+y^2}{1+y^2}} + C \]

Therefore

\[ \int_{c_1} \mathbf{F} \cdot d\mathbf{y} = f(3,4) - f(0,0) \]

\[ = \left( \ln \sqrt{\frac{1+3^2+4^2}{1+4^2}} + C \right) \]

\[ - \left( \ln \sqrt{\frac{1+0}{1}} + C \right) \]

\[ = \ln \sqrt{\frac{26}{17}} - \ln(1) = \ln \sqrt{\frac{26}{17}} \]
Green's Theorem (in plane)

Green's theorem gives the relationship between the line integral around a simple closed curve and the (double) integral over the region $D$ (in the plane) bounded by $C$.

\[ C = \int_D \]

A simple closed curve $C$ in the plane bounds a region/domain $D$ of this plane called the interior of $C$. This is part of the so-called Jordan Curve theorem.
Orientation of a curve: Let $C$ be a simple closed curve in the plane. We will usually orient $C$ in the counterclockwise direction. Then the region $D$ enclosed by the curve $C$ will always be on the left as one moves along the curve. Such a curve will be called positively oriented.

Whenever we speak of the area bounded by a simple closed curve $C$, we mean the area of the interior of $C$. 
Let $C$ be piecewise-differentiable, simple closed curve in the plane and let $D$ be the region bounded by $C$. Then for any vector field $\mathbf{F} = (P, Q)$ with continuous partial derivatives in $D$ one has

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Here the curve $C = \partial D$ is oriented anticlockwise.

This is also written as:

$$\int_P \, dx + \int_Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$
Example. \( D \) = "pie shaped region" in unit circle in the first quadrant.

\[
\oint_{C} \left( x^2+y^2 \right) \, dx + 2 \left( x^2+y^2 \right) \, dy
\]

\[
= \iint_{D} (4x - 2y) \, dA
\]

by Green's theorem. Let's verify this.

\[
\vec{F} = \left( x^2+y^2, \ 2 \left( x^2+y^2 \right) \right)
\]

\[
\frac{\partial P}{\partial y} = 2y \quad \text{and} \quad \frac{\partial Q}{\partial x} = 4x
\]

\[
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4x - 2y
\]
\[ C = C_1 \cup C_2 \cup C_3 \]

On,
\[ C_1 : \quad x^2 + y^2 = 1, \quad x \geq 0, \quad y \geq 0 \]
\[ C_2 : \quad x = 0, \quad 1 \leq y \leq 0 \]
\[ C_3 : \quad y = 0, \quad 0 \leq x \leq 1 \]

So we can parametrize them as:
\[ C_1 : \quad (\cos t, \sin t), \quad 0 \leq t \leq \pi/2 \]
\[ C_2 : \quad (0, 1-t), \quad 0 \leq t \leq 1 \]
\[ C_3 : \quad (t, 0), \quad 0 \leq t \leq 1 \]

Then
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} \]

\[ \begin{align*}
&= \int_0^{\pi/2} \left\langle (\cos^2 t + \sin^2 t, 2(\cos^2 t + \sin^2 t)), (-\sin t, \cos t) \right\rangle dt \\
&\quad + \int_0^1 \left\langle (t^2 + (1-t)^2, 2(t^2 + (1-t)^2)), (0, -1) \right\rangle dt \\
&\quad + \int_0^1 \left\langle (t^2 + 0^2, 2(t^2 + 0^2)), (1, 0) \right\rangle dt
\end{align*} \]
So, \[ \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{\pi/2} \left( \frac{d}{dt} \right) \mathbf{F}(t) \cdot d\mathbf{r} \]

\[ = \int_{0}^{\pi/2} \left( (1, 2), (\sin t, \cos t) \right) dt + \int_{0}^{1} \left( (1-t^2), 2(1-t^2), (0, -1) \right) dt \]

\[ + \int_{0}^{1} \left( (t^2), 2t^2), (1, 0) \right) dt \]

\[ = \int_{0}^{\pi/2} \left( -\sin t + 2\cos t \right) dt - 2 \int_{0}^{1} (1-t)^2 dt + \int_{0}^{1} t^2 dt \]

\[ = \left[ \cos t \right]_{0}^{\pi/2} + 2 \left[ \sin t \right]_{0}^{\pi/2} + 2 \int_{0}^{1} (1-t)^2 dt + \int_{0}^{1} t^2 dt \]

\[ = -1 + 2 + \frac{2}{3} \left( 0 - 1 \right) + \frac{1}{3} \left( 1 - 0 \right) \]

\[ = 1 - \frac{2}{3} + \frac{1}{3} = \frac{2}{3} \]
\[ \iint_D (4x - 2y) \, dA \]

using polar coordinates

\[ = \int_0^{\pi/2} \int_0^1 (4r \cos \theta - 2r \sin \theta) \, r \, dr \, d\theta \]

\[ = \int_0^{\pi/2} \left( \cos \theta \left[ \frac{4}{3} r^3 \right]_0^1 - \sin \theta \left[ \frac{2}{3} r^3 \right]_0^1 \right) \, d\theta \]

\[ = \int_0^{\pi/2} \left( \frac{4}{3} \cos \theta - \frac{2}{3} \sin \theta \right) \, d\theta \]

\[ = \frac{4}{3} \left[ \sin \theta \right]_0^{\pi/2} + \frac{2}{3} \left[ \cos \theta \right]_0^{\pi/2} \]

\[ = \frac{4}{3} + \frac{2}{3} (0 - 1) = \frac{4}{3} - \frac{2}{3} = \frac{2}{3} \]
So indeed

\[ \int_C F \cdot d\mathbf{r} = \iint_D (4x - 2y) \, dA \]

E.g., calculating the area of the domain/region $D$

bounded by curve $C = \partial D$.

The area of $D$ is:

\[ \text{Area}(D) = \iint_D dA \]

Green's theorem tells that:

\[ \int_{\partial D} P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \]
So if we take \( P \) and \( Q \) as:

\[
\frac{\partial P}{\partial y} = 0 \quad \text{and} \quad \frac{\partial Q}{\partial x} = 1 \quad \Rightarrow \quad Q = x + C
\]

Then

\[
\int_{D} x \, dy = \iint_{D} dA = \text{Area}(D)
\]

The if \( C = \partial D \) is parameterized as:

\[
\vec{r}(t) = (\gamma_1(t), \gamma_2(t)), \quad a \leq t \leq b
\]

then

\[
\int_{a}^{b} \gamma_1(t) \gamma_2'(t) \, dt = \text{Area}(D)
\]

Similarly if we take \( P \) & \( Q \) as:

\[
\frac{\partial P}{\partial y} = -1 \quad \Rightarrow \quad P = -y + C
\]

Then

\[
\int_{D} y \, dx = \iint_{D} dA = \text{Area}(D)
\]
Hence, \[ \text{Area}(D) = \frac{1}{2} \int_{D} -y \, dx + x \, dy \]

\[ = \frac{1}{2} \int_{D} \mathbf{F} \cdot d\mathbf{r} \]

where \( \mathbf{F} = (-y, x) \)

the spin vector field.

\[ \text{e.g.} \]

What is the area of the polygon having vertices: \( (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \)

\[ \text{D} \]
Sol: \[ \text{Area (D)} = \frac{1}{2} \int_C -y \, dx + x \, dy \]

\[ = \frac{1}{2} \sum_{i=1}^{n} \int_{c_i} -y \, dx + x \, dy \]

\( C_i \) is the segment from \((x_i, y_i)\) to \((x_{i+1}, y_{i+1})\) for \(1 \leq i \leq n-1\)

\( C_n \) is the segment from \((x_n, y_n)\) to \((x_1, y_1)\)

\[ C_i : \gamma_i(t) = (1-t) (x_i, y_i) + t (x_{i+1}, y_{i+1}) \quad , \quad 0 \leq t \leq 1 \]

for \(1 \leq i \leq n-1\)

\[ = \begin{pmatrix} (1-t) x_i + t x_{i+1} \\ (1-t) y_i + t y_{i+1} \end{pmatrix} \]

\[ \gamma_i'(t) = \begin{pmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{pmatrix} \]

Similarly \[ C_n : \gamma_n(t) = (1-t) (x_n, y_n) + t (x_1, y_1) \quad , \quad 0 \leq t \leq 1 \]

\[ = \begin{pmatrix} (1-t) x_n + t x_1 \\ (1-t) y_n + t y_1 \end{pmatrix} \]
and \[ \overrightarrow{y}_n'(t) = (x_n - x_n', y_n - y_n') \]

Then \[ \langle \vec{F}(\overrightarrow{x}(t)), \overrightarrow{y}_n'(t) \rangle \] where \[ \vec{F} = (-y, x) \]

\[ \left\langle \begin{array}{l} \left(1-t\right)y_i + ty_{i+1} \right\rangle + \left((1-t)x_i + tx_{i+1}\right) \right. \]

\[ \left( x_{i+1} - x_i \right) \overrightarrow{i} \right. + \left( y_{i+1} - y_i \right) \overrightarrow{j} \]

\[ \left( x_{i+1} - x_i \right) \overrightarrow{i} \right. + \left( y_{i+1} - y_i \right) \overrightarrow{j} \]

\[ = \left(1-t\right) \left( x_{i+1} - x_i \right) y_i + t \left( y_{i+1} - y_i \right) \left( x_{i+1} - x_i \right) \]

\[ + \left(1-t\right)x_i \left( y_{i+1} - y_i \right) \]

\[ + t \left( y_{i+1} - y_i \right) \]

\[ = \left(1-t\right) \left\{ \left( x_{i+1} - x_i \right)y_i + x_i \left( y_{i+1} - y_i \right) \right\} \]

\[ + t \left\{ -y_{i+1} \left( x_{i+1} - x_i \right) + x_{i+1} \left( y_{i+1} - y_i \right) \right\} \]
\[
= (1-t) \left\{ -y_i x_{i+1} + x_i y_i + x_{i+1} y_{i+1} - x_i y_{i+1} \right\} \\
+ t \left\{ -x_{i+1} y_{i+1} + x_i y_{i+1} + x_{i+1} y_{i+1} - x_{i+1} y_i \right\} \\
= (1-t) (x_i y_{i+1} - x_{i+1} y_i) + t (x_i y_{i+1} - x_{i+1} y_i) \\
= (1 - t + t) (x_i y_{i+1} - x_{i+1} y_i) \\
= x_i y_{i+1} - x_{i+1} y_i
\]

So
\[
\int_{C_i} -y \, dx + x \, dy = \int_0^1 \left< F(x_i(t), y_i(t)), z_i(t) \right> \, dt
\]
\[
= \int_0^1 (x_i y_{i+1} - x_{i+1} y_i) \, dt
\]
\[
\int_{C_i} -y \, dx + x \, dy = x_i y_{i+1} - x_{i+1} y_i
\]
Similarly, \[ \int -y \, dx + x \, dy = x_n y_1 - x_1 y_n \] 

Hence, \[ \text{Area}(D) = \frac{1}{2} \sum_{i=1}^{n} \int_{c_i} -y \, dx + x \, dy \]

\[ \text{Area}(D) = \frac{1}{2} \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i) \]

where \[ \begin{cases} x_{n+1} = x_1 \\ y_{n+1} = y_1 \end{cases} \]

E.g. Sometimes the double integral is easier to calculate.

Evaluate \[ \int_{C} F \cdot dr \]

where \( C \) is the circle of radius 2 centered at \((0,2)\) oriented clockwise.

\[ F = (y + e^{\sqrt{x}}, 2x + \cos(y^2)) \]
Functions like $e^{\sqrt{x}}$ and $\cos(y^2)$ are hard to integrate and doing the line integral by a parametrization of $C$ is awful!!

Instead, we will use the Green's theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

minus because $C$ has clockwise orientation.

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = - \iint_D \left( \frac{\partial}{\partial x} (2x + \cos(y^2)) - \frac{\partial}{\partial y} (y + e^{\sqrt{x}}) \right) \, dA$$

$$= - \iint_D (2 - 1) \, dA = - \iint_D 1 \, dA$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = - \text{Area}(D) = -4\pi$$
Green's theorem also works on domains/regions with holes as long as we orient the boundary correctly.

\[ \partial D = C_1 + C_2 + C_3 \]

- Outside loops get counter-clockwise orientation.
- Inside loops get clockwise orientation.

Then for vector field \( \vec{F} = (P, Q) \) with continuous partial derivatives:

\[
\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA
\]
Example: Careful! \( \vec{F} \) has to have continuous partial derivatives everywhere in \( D \), to apply
the Green's theorem.

Recall that for \( \vec{F} = \left( -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \)

\[ P = \frac{-y}{x^2+y^2}, \quad Q = \frac{x}{x^2+y^2} \]

Then

\[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \]

Hence

\[ \int_C \vec{F} \cdot d\vec{r} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \]

\[ \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} \quad \Rightarrow \quad \int_C \vec{F} \cdot d\vec{r} \]

But we have seen earlier

that

\[ \int_C \vec{F} \cdot d\vec{r} = 2\pi \]

\( \vec{F} \) is not continuous at \((0,0)\) and so is not defined everywhere in \( D \), therefore one cannot apply Green's theorem to \( \vec{F} \).
One can fix this by adding a small circle around \((0,0)\)

new \(D\) has a small circular hole around the origin.

\[\partial D = C + C'\]

Then we can apply the Green's theorem in \(D\).

So,

\[
\int \vec{F} \cdot d\vec{S} = \int \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

\[\partial D\]

\[\rightarrow \quad \int_{\partial D} \vec{F} \cdot d\vec{S} = 0\]

\[\rightarrow \quad \int_{C} \vec{F} \cdot d\vec{S} + \int_{C'} \vec{F} \cdot d\vec{S} = 0\]

\[\rightarrow \quad \int_{C} \vec{F} \cdot d\vec{S} = - \int_{C'} \vec{F} \cdot d\vec{S} = 2\pi \quad \text{(Calculated earlier)}\]
So for any curve $C$ around the origin

$$\int_C \vec{F} \cdot d\vec{r} = 2\pi \quad \text{(if $C$ goes around the origin once in the counterclockwise direction)}$$

E.g.- Sometimes, Green's theorem can be used to compute a hard integral by closing the curve.

Compute $\int_C \vec{F} \cdot d\vec{r}$ where $C$ is the part of the parabola $y = 1 - x^2$ from $(-1,0)$ to $(1,0)$.

$$\vec{F} = (1+y^2)\hat{i} + (x + \tan(e^{y^2}))\hat{j}$$
Solution: One has
\[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 2y \neq 0 \]
so \( \mathbf{F} \) is not conservative.

The curve \( C \) is not closed, so to apply the Green's theorem we add the curve \( C' \):

\[ C' : \text{segment from } (-1,0) \text{ to } (1,0) \]
\[ = (1-t)(-1,0) + t(1,0) \]
\[ = (2t-1,0), \quad 0 \leq t \leq 1 \]

On \( C' \), \( y = 0 \).

\[ \partial D = C - C' \]

By Green's theorem,
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{C'} \mathbf{F} \cdot d\mathbf{r} + \int_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_D (1-2y) \, d\mathcal{A} \]

not a very difficult function to integrate.
\[ \int_{C'} \mathbf{F} \cdot d\mathbf{r} = - \int_{D} (1 - 2y) \, dA + \int_{C} \mathbf{F} \cdot d\mathbf{r} \]

\[ = - \int_{\chi=-1}^{1} \left( \int_{y=0}^{1} (1 - 2y) \, dy \right) \, dx + \int_{0}^{1} (1; \ 2t-1+t\tan 1), (2o) \, dt \]

\[ = - \int_{\chi=-1}^{1} \left\{ (1 - x^2) - (1 - x^2)^2 \right\} \, dx + 2 \int_{0}^{1} dt \]

\[ = - \int_{\chi=-1}^{1} \left( x^2 - x^4 \right) \, dx + 2 = - \left[ \frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_{-1}^{1} + 2 \]

\[ = - \left[ \left( \frac{2}{3} - \frac{2}{5} \right) + 2 = \frac{-4}{15} + 2 \right. \]

**Therefore,** \[ \int_{C} \mathbf{F} \cdot d\mathbf{r} = \frac{26}{15} \]