1. (34 points) Let $\mu$ be fixed. Use the method of separation of variables to solve the following heat conduction boundary value problem:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \mu^2 u \quad \text{for } 0 < x < \pi/2 \text{ and } t > 0,
\]

Initial condition $u(x, 0) = \cos(3x)$ for $0 < x < \pi/2$,

Boundary condition $\frac{\partial u}{\partial x}(0, t) = 0, u(\pi/2, t) = 0$ for all $t > 0$.

Show all the cases when solving the appropriate eigenvalue problem.

2. (42 points) Solve the following heat conduction boundary value problem:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x \quad \text{for } 0 < x < \pi/2 \text{ and } t > 0,
\]

Initial condition $u(x, 0) = 1 + \sin(3x)$ for $0 < x < \pi/2$,

Boundary condition $u(0, t) = e^{-\mu t}, \frac{\partial u}{\partial x}(\pi/2, t) = t$ for all $t \geq 0$.

where $\mu > 1$.

(a) (4 points) Determine a function $w(x, t)$ satisfying the time-dependent boundary conditions.

(b) (8 points) Let

$$u(x, t) = v(x, t) + w(x, t)$$

Identify the PDE, the initial condition and the boundary conditions satisfied by $v(x, t)$.

(c) (30 points) Using appropriate eigenfunction expansion, solve the initial boundary value problem for $v(x, t)$ and from this determine the solution $u(x, t)$. 
Trigonometric Identities

\[
\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta; \quad \text{for } \alpha = \beta: \quad \sin(2\alpha) = 2 \sin \alpha \cos \alpha
\]

\[
\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta; \quad \text{for } \alpha = \beta: \quad \cos(2\alpha) = 2 \cos^2 \alpha - 1 = \cos^2 \alpha - \sin^2 \alpha
\]

A Summary of Eigenvalue Boundary Value Problems and their Eigenvalues and Eigenfunctions

\[Y'' + \lambda^2 Y = 0 \text{ in } (0, L)\]

Dirichlet B.C: \[Y(0) = 0 = Y(L) \implies \begin{cases} 
Y_n(x) = \sin(\lambda_n x) \\
\lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots
\end{cases}\]

Neumann B.C: \[Y'(0) = 0 = Y'(L) \implies \begin{cases} 
Y_n(x) = \cos(\lambda_n x) \\
\lambda_n = \frac{n\pi}{L}, \quad n = 0, 2, \ldots
\end{cases}\]

Mixed B.C: \[Y(0) = 0 = Y'(L) \implies \begin{cases} 
Y_n(x) = \sin(\lambda_n x) \\
\lambda_n = \frac{(2n+1)\pi}{2L}, \quad n = 0, 2, \ldots
\end{cases}\]

Mixed B.C: \[Y'(0) = 0 = Y(L) \implies \begin{cases} 
Y_n(x) = \cos(\lambda_n x) \\
\lambda_n = \frac{(2n+1)\pi}{2L}, \quad n = 0, 2, \ldots
\end{cases}\]

Periodic B.C: \[Y(0) = Y(L), Y'(0) = Y'(L) \implies \begin{cases} 
Y_n(x) \in \{1, \sin(\lambda_n x), \cos(\lambda_n x)\} \\
\lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots
\end{cases}\]

A Summary of guesses for \(w(x, t)\) to remove different non-homogeneous boundary conditions

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \gamma^2 u + F(x, t) \text{ for } 0 < x < L \text{ and } t > 0
\]

Let \(u(x, t) = v(x, t) + w(x, t)\)

Dirichlet B.C: \[u(0, t) = \varphi_1(t), \quad u(L, t) = \varphi_2(t); \quad w(x, t) = \varphi_1(t) + x \left( \frac{\varphi_2(t) - \varphi_1(t)}{L} \right)\]

Neumann B.C: \[\frac{\partial u}{\partial x}(0, t) = \varphi_1(t), \quad \frac{\partial u}{\partial x}(L, t) = \varphi_2(t); \quad w(x, t) = x \varphi_1(t) + x^2 \left( \frac{\varphi_2(t) - \varphi_1(t)}{2L} \right)\]

Mixed B.C: \[u(0, t) = \varphi_1(t), \quad \frac{\partial u}{\partial x}(L, t) = \varphi_2(t); \quad w(x, t) = \varphi_1(t) + x \varphi_2(t)\]

Mixed B.C: \[\frac{\partial u}{\partial x}(0, t) = \varphi_1(t), \quad u(L, t) = \varphi_2(t); \quad w(x, t) = (x - L) \varphi_1(t) + \varphi_2(t)\]