ASSIGNMENT 5

Laplace’s equation

Problem 1: Solve the Laplace’s equation in the semi-infinite strip

\[ D := \{(x,y) : 0 < x < \pi, y > 0\} \]

satisfying the following boundary conditions:

\[
\Delta u = 0 \quad \text{in} \quad D \\
\left\{ \begin{array}{l}
  u(0,y) = 0, \quad \frac{\partial u}{\partial x}(\pi,y) = 0 \quad \text{for} \quad y \geq 0, \\
  u(x,0) = \left\{ \\
    \begin{array}{ll}
      x & \text{for} \quad 0 \leq x \leq \pi/2 \\
      0 & \text{for} \quad \pi/2 < x \leq \pi \\
    \end{array}
  \right., \\
  \lim_{y \to +\infty} u(x,y) = 0.
\end{array} \right.
\]

Problem 2: Solve the Laplace’s equation in the semi-infinite strip

\[ D := \{(x,y) : 0 < x < 2, y > 0\} \]

satisfying the following boundary conditions:

\[
\Delta u = 0 \quad \text{in} \quad D \\
\left\{ \begin{array}{l}
  u(0,y) = 0, \quad \frac{\partial u}{\partial x}(2,y) = 0 \quad \text{for} \quad y \geq 0, \\
  u(x,0) = 2\sin\left(\frac{3\pi}{4}x\right) - 3\sin\left(\frac{7\pi}{4}x\right) \quad \text{for} \quad 0 \leq x \leq 2, \\
  \lim_{y \to +\infty} u(x,y) = 0.
\end{array} \right.
\]

Problem 3: Solve the Laplace’s equation in the rectangular region

\[ D := \{(x,y) : 0 < x < 1, 0 < y < 1\} \]

satisfying the following boundary conditions:

\[
\Delta u = 0 \quad \text{in} \quad D \\
\left\{ \begin{array}{l}
  \frac{\partial u}{\partial x}(0,y) = \beta(1 - \cos(2\pi y)), \quad u(1,y) = 0 \quad \text{for} \quad 0 \leq y \leq 1, \\
  \frac{\partial u}{\partial y}(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,1) = 0 \quad \text{for} \quad 0 \leq x \leq 1.
\end{array} \right.
\]

here \( \beta > 0 \) is a constant.

Problem 4: Solve the Laplace’s equation in the quarter-annular region

\[ D := \{(r,\theta) : 0 < a < r < b, \quad 0 < \theta < \pi/2\} \]
satisfying the following boundary conditions:

\[ \Delta u := \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{in } D \]

Boundary conditions \( \begin{cases} 
    u(r,0) = 0, & \frac{\partial u}{\partial \theta}(r, \pi/2) = 0 \quad \text{for } a \leq r \leq b, \\
    u(a, \theta) = \sin(3\theta), & u(b, \theta) = 0 \quad \text{for } 0 \leq \theta \leq \pi/2.
\end{cases} \)

**Problem 5:** Solve the Laplace’s equation in the semi-annular region

\[ D := \{ (r, \theta) : 0 < a < r < b, \ 0 < \theta < \pi \} \]

satisfying the following boundary conditions:

\[ \Delta u := \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{in } D \]

Boundary conditions \( \begin{cases} 
    \frac{\partial u}{\partial \theta}(r,0) = 0, & \frac{\partial u}{\partial \theta}(r, \pi) = 0 \quad \text{for } a \leq r \leq b, \\
    u(a, \theta) = 0, & u(b, \theta) = 1 + \cos(2\theta) \quad \text{for } 0 \leq \theta \leq \pi.
\end{cases} \)

**Problem 6:** Solve the Laplace’s equation in the semi-annular region

\[ D := \{ (r, \theta) : \pi < r < 2\pi, \ 0 < \theta < \pi \} \]

satisfying the following boundary conditions:

\[ \Delta u := \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{in } D \]

Boundary conditions \( \begin{cases} 
    u(r,0) = \sin(r), & u(r, \pi) = 0 \quad \text{for } \pi \leq r \leq 2\pi, \\
    u(\pi, \theta) = 0, & u(2\pi, \theta) = 0 \quad \text{for } 0 \leq \theta \leq \pi.
\end{cases} \)

**Problem 7:** Solve the Laplace’s equation in the external domain

\[ D := \{ (r, \theta) : r > 1, \ 0 < \theta < 2\pi \} \]

satisfying the Neumann boundary conditions:

\[ \Delta u := \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{in } D \]

Boundary conditions \( \begin{cases} 
    \frac{\partial u}{\partial \theta}(1, \theta) = f(\theta) \quad \text{for } 0 \leq \theta \leq 2\pi, \\
    \lim_{r \to +\infty} |u(r, \theta)| < +\infty.
\end{cases} \)
Strum-Liouville Problems

Problem 8: Consider the eigenvalue problem:

\[
\phi'' + 6\phi' + \lambda \phi = 0 \quad \text{for } 0 < x < L, \\
\phi(0) = 0, \, \phi(L) = 0.
\]

Reduce this problem to the form of Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions.

Problem 9: Determine the eigenvalues and corresponding eigenfunctions for the following boundary value problems:

(a) \[ y'' + \lambda y = 0 \quad \text{in } (0,1), \quad \text{with } y'(0) - y(0) = 0, \, y'(1) = 0. \]

(b) \[ y'' + \lambda y = 0 \quad \text{in } (0,1), \quad \text{with } y'(0) - y(0) = 0, \, y(1) + y'(1) = 0. \]

(c) \[(xy')' + x^{-1} \lambda y = 0 \quad \text{in } (a,b), \quad \text{with } y(a) = 0, y(b) = 0. \]

Problem 10: Consider the eigenvalue problem:

\[
x^2 y'' + xy' + \lambda y = 0 \quad \text{for } 1 < x < 2, \\
y(1) = 0, \, y'(2) = 0.
\]

(a) Reduce this problem to the form of Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions.

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace’s equation on the quarter-annular region

\[
D := \{(r, \theta) : 1 < r < 2, \, 0 < \theta < \pi/2\}
\]

satisfying the following boundary conditions:

\[
\Delta u := \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{in } D \\
\]

Boundary conditions \[
\begin{aligned}
u(r, 0) &= 0, \quad \frac{\partial u}{\partial r}(r, \pi/2) = f(r) \quad \text{for } 1 \leq r \leq 2, \
u(1, \theta) &= 0, \quad \frac{\partial u}{\partial r}(2, \theta) = 0 \quad \text{for } 0 \leq \theta \leq \pi/2.
\end{aligned}
\]

Problem 11: Consider the eigenvalue problem:

\[
-x^2 \left( \frac{\phi'}{x} \right)' = \lambda \frac{\phi}{x^3} \quad \text{for } 1 < x < 2, \\
\phi(1) = 0, \, \phi(2) = 0.
\]
(a) Determine the eigenvalues and corresponding eigenfunctions.

(b) Use the eigenfunctions in (a) to solve the following heat conduction boundary problem:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - x \frac{\partial u}{\partial x} \quad \text{for } 1 < x < 2 \text{ and } t > 0,
\]

Initial condition \( u(x, 0) = x \) for \( 1 < x < 2 \),

Boundary condition \( u(1, t) = 0, u(2, t) = 0 \) for all \( t \geq 0 \).