Heat equation, Separation of variables and Fourier expansion

Problem 1: Find all eigenvalues and corresponding eigenfunctions for the following problems
(i) \( y'' + \lambda y = 0 \) for \( 0 < x < 1 \), with \( y(0) = 0, y'(1) = 0 \).
(ii) \( x^2 y'' + xy' + \lambda y = 0 \) for \( 1 < x < 2 \), with \( y(1) = 0, y(2) = 0 \). Only consider the case \( \lambda > 0 \).

Problem 2: Find the Full Fourier series expansion of the function \( f(x) = x^2 \) for \( -1 < x < 1 \) which has a period 2. Use this series to determine a series expansion for \( \frac{\pi^2}{6} \).

Problem 3: Use the method of separation of variables to solve the following boundary value problem for the heat equation
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad t > 0,
\]
Initial condition \( u(x,0) = \cos \left( \frac{3\pi x}{2} \right) \) for \( 0 < x < 1 \),
Boundary condition \( \frac{\partial u}{\partial x}(0,t) = 0, u(1,t) = 0 \) for all \( t \geq 0 \).

Problem 4: Let \( k \) be a fixed positive integer. Use the method of separation of variables to solve the following Neumann boundary value problem for the heat equation
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 < x < \pi \quad \text{and} \quad t > 0,
\]
Initial condition \( u(x,0) = \cos (kx) \) for \( 0 \leq x \leq \pi \),
Boundary condition \( \frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(\pi,t) = 0 \) for all \( t > 0 \).
**Problem 5:** Use the method of separation of variables to solve the following linear parabolic boundary value problem

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u \quad \text{for } 0 < x < 1 \text{ and } t > 0,
\]

Initial condition \( u(x, 0) = \cos (\pi x) \) for \( 0 < x < 1 \),

Boundary condition \( u(0, t) = 0, u(1, t) = 0 \) for all \( t \geq 0 \).

Then find \( \lim_{t \to +\infty} u(x, t) \).

*Hint:* 
\[
\int_0^1 \sin(n\pi x) \cos(\pi x) = \begin{cases} 
0 & \text{if } n \text{ is odd}, \\
\frac{2n}{n^2-1} & \text{if } n \text{ is even}.
\end{cases}
\]

**Problem 6:** Consider \( f(x) = x(\pi - x) \) for \( 0 \leq x \leq \pi \). Let \( \tilde{f}(x) \) be the odd periodic extension of period \( 2\pi \) of \( f(x) \).

(i) Determine the Fourier series for the function \( \tilde{f}(x) \).

(ii) Use the Fourier series to show that

\[
\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots
\]

(iii) Use the Parseval’s theorem to show that

\[
\frac{\pi^6}{960} = \sum_{n=1}^{+\infty} \frac{1}{(2n - 1)^6}
\]