Chapter 13: Cardinality of Sets:

In this chapter we are going to talk about the sizes of sets. We saw that for finite sets, the size means the number of elements of the set, e.g. 
A = \{a, b, c, d, e, f\} \Rightarrow |A| = 6, which sounds quite trivial & easy to understand, "just count the number of elements." But what happens when we have an infinite set like \(\mathbb{N}\)? Can we still talk about the size of \(\mathbb{N}\)? If so, can we compare it to the size of \(\mathbb{Z}\) or \(\mathbb{Q}\) or \(\mathcal{P}(\mathbb{N})\)? In this chapter, we are going to answer these questions.

13.1: Sets with equal Cardinalities:

Definition: Two sets \(A\) and \(B\) have the same cardinality, written \(|A| = |B|\), if there exists a bijective function \(f: A \rightarrow B\). If no such bijective function exists, then the sets have unequal cardinalities, that is \(|A| \neq |B|\).

Example: If \(A\) & \(B\) are finite and their sizes are the same, then their cardinalities are the same. Say, if \(|A| = 5 = |B|\), then \(f: A \rightarrow B\), defined as below

\[
A = \{a_1, a_2, a_3, a_4, a_5\} \quad \text{and} \quad B = \{b_1, b_2, b_3, b_4, b_5\}
\]

is a bijection.

\[
f = \begin{pmatrix}
a_1 & \rightarrow & b_1 \\
a_2 & \rightarrow & b_2 \\
a_3 & \rightarrow & b_3 \\
a_4 & \rightarrow & b_4 \\
a_5 & \rightarrow & b_5 \\
\end{pmatrix}
\]
Here we need to be a little careful. Given two sets $A$ and $B$ and a function $f: A \rightarrow B$, just because the given $f$ is not bijective does not mean that $|A| \neq |B|$. For example:

$f: A \rightarrow B$ is not bijective, but still $|A| = |B|$ since we can write the function $g: A \rightarrow B, g = \{(1, a), (2, b), (3, c), (4, d)\}$ which is a bijective function.

For general finite sets of size $n$, we can use induction. Namely, we can prove:

**Proposition:** Two finite sets of the same size have the same cardinality.

**Proof:** (By induction): **Basis step:** If $|A| = |B| = 1$, then calling the elements of $A$ and $B$, $a$ and $b$ respectively, we can define $f: A \rightarrow B$ s.t. $f(a) = b$. Thus $f$ is bijective, thus $|A| = |B|$. 

**Inductive step:** Assume sets of size $n$ have the same cardinality. Then we want to show that two set of size $n+1$ have the same cardinality.

Let $A$ and $B$ be sets s.t. $|A| = n+1 = |B|$. Then
Take $\tilde{A}$ & $\tilde{B}$, subsets of $A$ & $B$ s.t $|\tilde{A}| = n = |\tilde{B}|$. Then by inductive hypothesis, we know that there is a bijection (i.e. bijective function) from $\tilde{A}$ to $\tilde{B}$, call it $f$. Then we see $|A - \tilde{A}| = 1 = |B - \tilde{B}|$. Call $a \in \tilde{A} - \tilde{A}$ & $b \in \tilde{B} - \tilde{B}$. Then we see $b \not\in f(\tilde{A})$ & $a \not\in f^{-1}(\tilde{B})$. Thus let $f: A \to B$ s.t. $f(a) = b$ & $f(\tilde{A}) = f(\tilde{B})$ for any $\tilde{A} \in \tilde{A}$, then $f$ is a bijection, which finishes the proof.

Now, we can have fun with more complicated examples.

Example: Show $|\mathbb{N}| = |\mathbb{Z}|$.

For this, let's take the function $f: \mathbb{N} \to \mathbb{Z}$ defined as

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(n)$</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>-2</td>
<td>3</td>
<td>-3</td>
<td>4</td>
<td>-4</td>
<td>5</td>
<td>-5</td>
<td>6</td>
<td>-6</td>
<td>…</td>
</tr>
</tbody>
</table>

Then we see that $f$ is surjective since every integer appears in the image of $f$ & also every integer appears only once i.e. $f$ is injective. Thus $f$ is a bijection, meaning $|\mathbb{N}| = |\mathbb{Z}|$.

We can actually write the explicit form of this function:

$f: \mathbb{N} \to \mathbb{Z}$ s.t. $f(n) = \begin{cases} k & \text{if } n = 2k \\ -k & \text{if } n = 2k+1, \end{cases}$
and we can check that $f$ is actually bijective. Therefore, even though $\mathbb{N} \times 2$ are both infinite sets, we see that they have the same “size” of infinity.

Now, you may be asking, “doesn’t this mean that they are just infinite sets? So the “size” is “infinity”?” The answer is: “Not really. There are different “sizes” of “infinity.”” Let’s see that not all infinities are the same.

**Theorem:** There is no bijection between $\mathbb{N} \times 12$.
Therefore $|\mathbb{N}| \neq |\mathbb{R}|$.

This first was realized by Georg Cantor & to this day we still use his ingenious idea called Cantor’s diagonal argument.

**Proof:** Assume for a contradiction that there is a bijection from $\mathbb{N}$ to $\mathbb{R}$, call it $f$.

Then we know that there is a (possibly infinite) decimal expansion for any real number. Now if we make a table for $f$, we get
<table>
<thead>
<tr>
<th>n</th>
<th>f(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(a_1, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, 16, 17, 18)</td>
</tr>
<tr>
<td>2</td>
<td>(a_2, a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, 26, 27, 28)</td>
</tr>
<tr>
<td>3</td>
<td>(a_3, a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, 36, 37, 38)</td>
</tr>
<tr>
<td>4</td>
<td>(a_{41}, a_{41}, a_{42}, a_{43}, a_{44}, a_{45}, a_{46}, 47, 48)</td>
</tr>
<tr>
<td>5</td>
<td>(a_5, a_{51}, a_{52}, a_{53}, a_{54}, a_{55}, a_{56}, 57, 58)</td>
</tr>
<tr>
<td>6</td>
<td>(a_6, a_{61}, a_{62}, a_{63}, a_{64}, a_{65}, a_{66}, 67, 68)</td>
</tr>
<tr>
<td>7</td>
<td>(a_{71}, a_{71}, a_{72}, a_{73}, a_{74}, a_{75}, a_{76}, 77, 78)</td>
</tr>
</tbody>
</table>

where \(a_i\)’s \& \(a_{ij}\)’s are in \(\mathbb{N}\) for all \(i \& j \in \mathbb{N}\).

Now look at the diagonal elements starting \(a_{11}\).

<table>
<thead>
<tr>
<th>n</th>
<th>f(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(a_1, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, 16, 17, 18)</td>
</tr>
<tr>
<td>2</td>
<td>(a_2, a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, 26, 27, 28)</td>
</tr>
<tr>
<td>3</td>
<td>(a_3, a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, 36, 37, 38)</td>
</tr>
<tr>
<td>4</td>
<td>(a_{41}, a_{41}, a_{42}, a_{43}, a_{44}, a_{45}, a_{46}, 47, 48)</td>
</tr>
<tr>
<td>5</td>
<td>(a_5, a_{51}, a_{52}, a_{53}, a_{54}, a_{55}, a_{56}, 57, 58)</td>
</tr>
<tr>
<td>6</td>
<td>(a_6, a_{61}, a_{62}, a_{63}, a_{64}, a_{65}, a_{66}, 67, 68)</td>
</tr>
<tr>
<td>7</td>
<td>(a_{71}, a_{71}, a_{72}, a_{73}, a_{74}, a_{75}, a_{76}, 77, 78)</td>
</tr>
</tbody>
</table>

Since we assumed that \(f\) is bijective, this table should cover every real number. We will now get a contradiction by writing a real number that is
not in the image of \( f \), using the red diagonal numbers. The construction goes as follows.

Let \( x = 0. b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 \ldots \) and

\[
\begin{align*}
b_1 &= 1 \text{ if } a_{11} \neq 1 \text{ and } b_1 = 2 \text{ if } a_{11} = 1, \\
b_2 &= 1 \text{ if } a_{22} \neq 1 \text{ and } b_2 = 2 \text{ if } a_{22} = 1, \\
b_3 &= 1 \text{ if } a_{33} \neq 1 \text{ and } b_3 = 2 \text{ if } a_{33} = 1, \\
&\vdots
\end{align*}
\]

Going on this way, we see that \( x \notin \mathbb{R} \), but

\[
\begin{align*}
x &\neq f(1) \text{ since } b_1 \neq a_{11}, \\
x &\neq f(2) \text{ since } b_2 \neq a_{22}, \\
x &\neq f(3) \text{ since } b_3 \neq a_{33}, \\
x &\neq f(4) \text{ since } b_4 \neq a_{44}, \\
&\vdots
\end{align*}
\]

Thus, \( x \notin f(\mathbb{N}) \) which contradicts with \( f \) being a bijection. Therefore, there cannot be any bijection between \( \mathbb{N} \) and \( \mathbb{R} \) (actually, this proof says, there cannot be any surjection from \( \mathbb{N} \) to \( \mathbb{R} \). Thus, \( |\mathbb{N}| \neq |\mathbb{R}| \).

Now, there is another nontrivial example.

**Example:** Show that \( |(0, \infty)| = |(0,1)| \).
To show that we need to find a bijection from \((0, \infty)\) to \((0,1)\) or from \((0,1)\) to \((0,\infty)\).

Consider the function defined as follows:

\[ f(x) = \frac{x}{x+1} \]

for every \(x \in \mathbb{R}\), we connect the points \( P = (-1,1) \) & \( x \) and we define \( f(x) \) to be the y-intercept \( x \) of this line.

We can see that this func \( f \) can be given as \( f(x) = \frac{x}{x+1} \) using similar triangles red & green.

But then we see that if \( y \in (0,1) \), then for \( x = \frac{y}{1-y} \), we get \( f(x) = \frac{x}{x+1} = (\frac{y}{1-y}) \cdot \frac{1}{1+1} = y. \)

i.e. \( f \) is surjective.

Moreover if \( f(x_1) = f(x_2) \), then \( \frac{x_1}{x_1+1} = \frac{x_2}{x_2+1} \) i.e \( \frac{x_1}{x_1+1} = \frac{x_2}{x_2+1} \rightarrow x_1 = x_2 \), which means \( f \) is injective.

Therefore \( f \) is bijective i.e. \( |(0, \infty)| = |(0,1)|. \)

Example: Show \( |\mathbb{R}| = |(0,1)|. \)

For this example, we are first going to show \( |\mathbb{R}| = |(0, \infty)| \) & using the bijection above.
between \((0,\infty)\) \& \((0,1)\), we will conclude \(|\mathbb{R}| = |(0,1)|\).

Consider \(g: \mathbb{R} \to (0,\infty)\) \st\( g(x) = e^x\). Then we know that since \(g\) is bijective, \(|\mathbb{R}| = |(0,\infty)|\).

Thus take \(f \circ g : \mathbb{R} \to (0,1)\). Since it is the composition of two bijective fncs, it is bijective. This means \(|\mathbb{R}| = |(0,1)|\).

\textbf{Note:} We can easily show that "has the same cardinality as" is an equivalence relation.

13.2: Countable \& Uncountable Sets:

We saw in the previous section that \(\mathbb{Z}\) \& \(\mathbb{N}\) have the same "size", but \(\mathbb{N}\) \& \(\mathbb{R}\) don't. So, to distinguish different size of infinities, we are going to define:

\textbf{Definition:} Suppose \(A\) is a set. Then \(A\) is countably infinite if \(|\mathbb{N}| = |A|\). The set is called \textit{uncountable} if \(A\) is infinite \& \(|\mathbb{N}| \neq |A|\).

Then we see that \(\mathbb{Z}\) is countable whereas \(\mathbb{N}\) or \((0,\infty)\) or \((0,1)\) are not.

\textbf{Definition:} The cardinality of the natural numbers is denoted by \(\mathcal{N}_0\). That is \(|\mathbb{N}| = \mathcal{N}_0\). Thus, any countable set has cardinality \(\mathcal{N}_0\).
Theorem: A set $A$ is countably infinite if and only if its elements can be arranged in an infinite list $a_1, a_2, a_3, \ldots$.

Proof: The proof of this theorem is quite straightforward. $A$ is countable iff $|\mathbb{N}| = |A|$, i.e.

iff there is a bijection $f : \mathbb{N} \to A$.

iff there is a bijection s.t. $f(\mathbb{N}) = A$.

iff $A = \{f(a_1), f(a_2), f(a_3), \ldots \}$, for some bijection, $f : a_1 \leftrightarrow a_2 \leftrightarrow a_3 \ldots$.

Although this theorem is quite trivial, it is not always easy to list the elements of a countably infinite set. As an example, we can see:

Theorem: The set $\mathbb{Q}$ of rational numbers is countably infinite.

Proof: To prove this we are going to use the previous theorem & list the elements of $\mathbb{Q}$.

This, as we can guess is not a trivial process. We are going to write the elements of $\mathbb{Z}$ in order and for every element, we are going to write the non-repeating rational numbers with the numerator being the integer given. Namely
Then, once we have the table above (much better than me drawing it...), we can arrange the terms simply by the process.

```
| 0 | 1 | -1 | 2 | -2 | 3 | -3 | 4 | -4 | 5 | -5 | ...
|---|---|----|---|----|---|----|---|----|---|----|---|
| 0 | 1 | -1 | 2 | -2 | 3 | -3 | 4 | -4 | 5 | -5 | ...
| 1 | 1 | -1 | 1 | 1 | 2 | -2 | 1 | 3 | -3 | 1 | 4 | -4 | 1 | 5 | -5 | 1 | ...
| 1 | 2 | -2 | 2 | 3 | -2 | 3 | 2 | -3 | 4 | -3 | 5 | 3 | -3 | 5 | 2 | -5 | 2 | ...
| 1 | 3 | -1 | 3 | 5 | -2 | 5 | 4 | -3 | 4 | -4 | 5 | 3 | -5 | 3 | 4 | -5 | 4 | ...
| 1 | 4 | -1 | 4 | 7 | -2 | 7 | 5 | -3 | 4 | -4 | 5 | 7 | -5 | 7 | 4 | -5 | 4 | ...
| 1 | 5 | -1 | 5 | 9 | -2 | 9 | 7 | -3 | 4 | -4 | 5 | 9 | -5 | 9 | 6 | -5 | 6 | ...
| 1 | 6 | -1 | 6 | 11 | -2 | 11 | 8 | -3 | 4 | -4 | 5 | 11 | -5 | 11 | 7 | -5 | 7 | ...
| 1 | 7 | -1 | 7 | 13 | -2 | 13 | 10 | -3 | 4 | -4 | 5 | 13 | -5 | 13 | 8 | -5 | 8 | ...
```
This means, we can arrange elements of \( \mathbb{Q} \) as 0, 1, \( \frac{1}{2} \), \( -\frac{1}{2} \), \( -1, 2 \), \( \frac{2}{3} \), \( \frac{2}{5} \), \( -\frac{2}{5} \), \( \frac{2}{7} \), \( -\frac{2}{7} \), \( \frac{1}{3} \), \( -\frac{1}{3} \), \( \frac{1}{4} \), \( -\frac{1}{4} \), \( \frac{1}{7} \), \( -\frac{1}{7} \), \ldots.

Thus, since the elements of \( \mathbb{Q} \) can be arranged, we see that \( |\mathbb{Q}| = |\mathbb{N}| \).

Theorem: If \( A \& B \) are countably infinite, then so is \( A \times B \).

Proof: Suppose \( A \& B \) are countably infinite, then we know that we can write \( A \& B \) in list form as

\[
A = \{a_1, a_2, a_3, \ldots \} \quad \& \\
B = \{b_1, b_2, b_3, \ldots \}
\]

Now we can use almost exactly the same argument in the previous proof. Namely we can make a table of elements of \( A \times B \) and list them as we listed elements of \( \mathbb{Q} \).
**Corollary**: Given $n$ countably infinite sets $A_1, A_2, A_3, \ldots, A_n$ with $n \geq 2$, the Cartesian product $(A_1 \times A_2 \times A_3 \times \ldots \times A_n)$ is also countably infinite.

**Proof**: The proof uses induction.

**Basis step**: ($n=2$) If $A_1$ & $A_2$ are countable, we see that $A_1 \times A_2$ is countable in the previous theorem.

**Inductive step**: Assume that $A_1, A_2, \ldots, A_n$ & $A_{n+1}$ are countable. Also assume $A_1 \times A_2 \times \ldots \times A_n$ is countable. Then we want to show $A_1 \times A_2 \times \ldots \times A_n \times A_{n+1}$ is countable. We see that, since $A_1 \times A_2 \times \ldots \times A_n$ is countable & $A_{n+1}$ is countable $(A_1 \times A_2 \times \ldots \times A_n) \times A_{n+1}$ is countable. Moreover, since

$$f: A_1 \times A_2 \times A_3 \times \ldots \times A_n \times A_{n+1} \rightarrow (A_1 \times A_2 \times \ldots \times A_n) \times A_{n+1}$$

$$f((a_1, a_2, \ldots, a_n, a_{n+1}) = ((a_1, a_2, \ldots, a_n), a_{n+1})$$

is a bijection, we get

$$|A_1 \times A_2 \times \ldots \times A_{n+1}| = (|A_1 \times A_2 \times \ldots \times A_n|) \times |A_{n+1}| = 1 \cdot IN.$$

**Theorem**: If $A$ & $B$ are countable & $A \cap B = \emptyset$, then $A \cup B$ is also countable.

**Proof**: The proof is going to use the argument of listing. Since $A$ & $B$ are countable, we
can write $A$ and $B$ as lists,
\[ A = \{a_1, a_2, a_3, \ldots \} \quad \& \quad B = \{b_1, b_2, b_3, \ldots \} \]
Then we can list the elements of $A \cup B$ simply as
\[ A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \ldots \} \]
Thus, $A \cup B$ is countable.

**Theorem:** An infinite subset of a countably infinite set is countably infinite.

**Corollary:** If $A \cup B$ is uncountable, then $A$ is uncountable. (As you can see, this is the contrapositive of the theorem)

**Proof of Theorem:** If $A$ is countably infinite, then we know that we can write $A$ as a list
\[ A = \{a_1, a_2, a_3, a_4, \ldots \} \]
Then if $B \subseteq A$, we can write $B$ as a list, by choosing the elements of $A$ in order in the list that are in $B$. Thus, $B$ is countably infinite.

13.3: Comparing Cardinalities:

Recall that the pigeonhole principle says that if we have two finite sets $A$ and $B$ s.t.
\[ |A| < |B| \]
then we cannot have a surjection
\[ f : A \to B \]
Moreover, if $A$ and $B$ are
\(|A| > |B|\), then we cannot have an injection 
\(g: A \rightarrow B\).

These are nice when we work with finite sets. We can, however, extend these ideas to infinite sets too.

**Definition**: Suppose \(A\) & \(B\) are sets.

1. \(|A| = |B|\) means there is a bijection from \(A\) to \(B\).
2. \(|A| < |B|\) means there is an injection from \(A\) to \(B\), but no surjection.
3. \(|A| \leq |B|\) means \(|A| < |B|\) or \(|A| = |B|\).

For example, we see that \(f: \mathbb{N} \rightarrow \mathbb{R}\), defined as \(f(n) = n\) is an injection, but we also saw before that there can be no surjection from \(\mathbb{N}\) to \(\mathbb{R}\) (when we proved \(|\mathbb{N}| \neq |\mathbb{R}|\)). Therefore \(|\mathbb{N}| < |\mathbb{R}|\). However, \(\mathbb{R}\) is not the "biggest" set we can construct. For example:

**Theorem**: If \(A\) is any set, then \(|A| < |\mathcal{P}(A)|\)

(which implies we can always construct "bigger" sets by taking power sets, e.g. \(|\mathbb{R}| < |\mathcal{P}(\mathbb{R})|\).)

**Proof**: If \(A\) is finite, this statement is clear, since \(\forall x \in \mathbb{R}, \ x \leq 2^x \&\)

\(|A| = \text{size of } A < 2^{\text{size of } A} = 2^{\left|A\right|} = |\mathcal{P}(A)|\).
Assume $A$ is any infinite set. We need to show that there is an injection from $A$ to $B$, but no surjection.

Finding injection is quite straightforward:

Let $f : A \to P(A)$ defined as $f(x) = \{x\}$, i.e. $f$ is the function that sends every element $x$ to the subset of $A$ that has only $x$ as an element. Thus, clearly $f$ is injective.

Now we need to show there is no surjection. To do that, assume for a contradiction that there is a surjection $f : A \to P(A)$. Then, this means for every element $x \in A$, $f(x)$ is a subset of $A$.

Then we see that either $x \in f(x)$ or $x \notin f(x)$.

Now define $B = \{x \in A : x \notin f(x)\} \subseteq A$. Since $B \subseteq A$, we see that $B \in P(A)$ & since $f$ is surjective, $B = f(a)$ for some $a \in A$. Then we have two options, $a \in B$ or $a \notin B$.

Case 1: $a \in B$. Then by definition of $B$, $a \notin f(a)$ & $a \in B$, which is a contradiction.

Case 2: $a \notin B$. Then by definition of $B$, $a \in f(a)$ & since $f(a) = B$, $a \notin B$, which is a contradiction.

Thus, there cannot be any surjection from
A to \mathbb{P}(A). Therefore \ |A| \leq |\mathbb{P}(A)|.

This is the end of the lectures.
Thank you all for your attention & for making teaching this class a joyful experience.
Good luck to you all with your finals & your future studies.