Chapter 2: Logic

In mathematics, we have axioms and using the axioms we have, we prove "facts" such as lemmas, theorems etc. To be able to do that we need a way to "combine" the facts we have, while keeping their truth value. This "way" is called logic. Using logic, we will look at how sentences "combine" to make another sentence and we will also be able to understand how this new sentence is related to its individual pieces.

2.1: Statements.

**Defn:** A statement is a sentence or a mathematical expression that is either definitely true or definitely false. Thus, statements are going to be the "sentences" we talked about that we are going to apply logic to produce other information.

**Ex:** If a circle has radius \( r \), then its area is \( \pi r^2 \) square units. (true statement)

- \( 2 \in \mathbb{Z} \) (reads as: 2 is an element of the set of integers) (true statement)
- $5=2$ (reads as: 5 is equal to 2) (false statement.)

- Some right triangles are equilateral (false statement)

- Add 5 to both sides (not a statement, since we cannot talk about whether or not this sentence is true or not. But, if we say "adding 5 to both sides of $x-5=37$ gives $x=42" is a statement.

We said that a sentence is a statement if it is true or false. This means that if a sentence has to be either true or false, it still is a statement even when we don’t know whether or not it is true or false. This may sound a bit confusing, but if we consider the example, it will become more clear:

"Every even integer greater than 2 can be written as the sum of two primes."

This sentence has to be true or false, although, as of this moment, no one knows whether it is true or not, so this sentence is a
statement (in fact, it is quite a famous statement, and is called "Goldbach’s conjecture". In fact, a conjecture is a statement claimed to be true, although we don’t know whether it is or not.)

In logic, when we work with statements, it is often practical to use letters for specific statements. This way we can avoid writing the same sentence over and over again.

\[ \text{ex: } P: \text{For every integer } n > 1, \text{ the number } 2^n - 1 \text{ is prime. (False statement)} \]

\[ Q: \text{Every polynomial of degree } n \text{ has at most } n \text{ roots (true statement).} \]

Here, as we did with the sets, we can use indices to denote different statements.

\[ \text{ex: } S_1: \mathbb{Z} \subseteq \emptyset \text{ (reads as: set of integers is a subset of empty set) (false statement)} \]

\[ S_2: \{0, -1, -2\} \cap \mathbb{N} = \emptyset \text{ (reads as: the intersection of the set \{0, -1, -2\} and \mathbb{N} is empty) (true statement)} \]
The statements can contain variables, but not every sentence that contains variables are statements.

ex: P: “If an integer $x$ is a multiple of 6, then $x$ is even” is a true statement whereas

Q: The integer $x$ is even” is not a statement since the “truth value” of this sentence depends on what “$x$” is.

A sentence, whose truth (“truth value”) depends on the value of one or more variables, is called an “open sentence”.

2.2 And, or, not:

We can use the words “and” and “or” to combine two statements to form a new statement.

ex: P: The number 8 is even and a power of 2.

can be seen as the combination of the two statements

R: The number 8 is even

Q: 8 is a power of 2. with the word “and”. Since we are using letters to denote sentences, we are also going to introduce new
notation to replace "and", "or" and "not".

To denote P, i.e. R "and" Q we write R \land Q. Here \( \land \) denote the word "and".

We see that the statement R \land Q is true if both R and Q are true,
otherwise it is false. We can write down all the possible "truth values" of this statement in, what is called, a "truth table".

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \land Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<td>T</td>
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</tbody>
</table>

In this table, T stands for "True", F stands for "False".

We can also combine two statements with "or".

*ex:* The statement

\( P: \text{"The number 7 is prime or 18 is odd"} \)

is the combination of the statements

\( Q: \text{The number 7 is prime} \)

R: The number 18 is odd.

with the word "or".
We will denote the statements of the form \( Q \lor R \) as \( Q \lor R \), i.e. \( \lor \) stands for "or". Here, when we say \( Q \lor R \), we mean "one or both of \( Q \) and \( R \). We see that such a statement can only be false when both \( Q \) and \( R \) are false, i.e., if we put it in a truth table, we get

\[
\begin{array}{ccc}
 P & Q & P \lor Q \\
 T & T & T \\
 T & F & T \\
 F & T & T \\
 F & F & F \\
\end{array}
\]

If we ever need to express the fact that only one of \( Q \) and \( R \) is true, we use one of the following:

"\( Q \) or \( R \), but not both"

"Either \( Q \) or \( R \)"

"Exactly one of \( Q \) or \( R \)."

Moreover, given any statement \( P \), we can form a new statement "it is not true that \( P \)." For example if we have the statement "\( 2 \in \emptyset \)" we can form "it is not true that \( 2 \in \emptyset \)," or from "\( 2 \) is even", we can form "it is not true that \( 2 \) is even" (which, by the way, is a false statement, where we see that the statement "\( 2 \) is even" is true).
We use the symbol \( \neg \) (or \( ! \) or \( \top \)) to denote "It's not true that", so \( \neg P \) (or \( !P \) or \( T_P \)) means "It is not true that \( P \)" which we often read as "not \( P \)."

And the truth table for \( \neg P \) is as follows:

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<tr>
<th>( P )</th>
<th>( \neg P )</th>
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</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( F )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
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</tbody>
</table>

The statement \( \neg P \) is called the negation of \( P \).

2.3 **Conditional statements.**

Given any two statements \( A \) and \( B \), we can form a new statement \( P \) as "If \( A \) then \( B \)." This is written symbolically as \( A \implies B \) which reads as "If \( A \) then \( B \)" or " \( A \) implies \( B \)." We say that this statement is false if it fails to deliver its claim, namely, \( A \implies B \) is false when \( A \) is true but yet \( B \) is false.

Such statements are called conditional statements. Their truth table is as follows:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \implies Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
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<tr>
<td>( T )</td>
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<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
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</tbody>
</table>

Here we see that when \( P \) is false, then \( P \implies Q \) is automatically true. The reason is that when \( P \) is false,
we cannot verify that $P \Rightarrow Q$ is false. (You can see this as, "every if-then statement is true unless proven false.")

Remark: When we say $P \Rightarrow Q$ is true or false, we are talking about the truth of the statement $P \Rightarrow Q$, not the truth of the outcome, or "then" part of the "if-then" statement, i.e., $Q$.

**Ex:** If it rains, then the roads get wet.

<table>
<thead>
<tr>
<th>it rains</th>
<th>roads get wet</th>
<th>$P \Rightarrow Q$ roads get wet</th>
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<tbody>
<tr>
<td>$T$</td>
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</table>

If you look at the last two rows on the truth table, we see that the statement "it rains $\Rightarrow$ roads get wet" is true independent of the truth value of the statement "roads get wet" when it doesn't rain.

There are different expressions that all mean "If $P$ then $Q$". Here are some of them:

- If $P$, then $Q$
- $Q$ if $P$
- $Q$, whenever $P$
- $Q$, provided $P$
- Whenever $P$, then also $Q$
- $P$ is a sufficient condition for $Q$
- $P$, it is necessary that $Q$
- For $Q$, it is sufficient that $P$
- $Q$, a necessary condition for $P$
Note: Since a conditional statement $P \Rightarrow Q$ can only be false when $P$ is true and $Q$ is false, to check the truth value of such a statement we can simply assume that $P$ is true is true if $P$ is an open sentence.

ex: If a function is continuous then it is differentiable.

Although the sentence "a function is continuous" is an open sentence (i.e. truth value of this sentence depends on the function we choose), we can assume that we are given any continuous function and we need to show that it is differentiable. This means that this statement should hold for every continuous function. So the question is, is it true that every continuous function is differentiable? The answer is NO! Since $f(x) = |x|$ is a continuous function but it is not differentiable.

2.4 Biconditional Statements:

We saw that the statements $P \Rightarrow Q$ is not the same as $Q \Rightarrow P$ (the converse of the statement). We saw that by showing that $P \Rightarrow Q$ and $Q \Rightarrow P$ have different truth tables. But sometimes it is going to happen that $P \Rightarrow Q$ and $Q \Rightarrow P$ will be true. Which means $(P \Rightarrow Q) \land (Q \Rightarrow P)$ is true. Thus, $P$ is both a necessary and a sufficient condition for $Q$. We will denote $(P \Rightarrow Q) \land (Q \Rightarrow P)$ as $P \Leftrightarrow Q$, which reads as "$P$ if and only if $Q$". We sometimes write this as "$P$ iff $Q$".
ex: A triangle is equilateral iff all its inner angles are 60°.
which means A triangle is equilateral if all its inner angles are 60°.

and

If a triangle is equilateral, then all its inner angles are 60°.

We already know that this statement is true, but let's analyse it in details.

Given a triangle $\triangle ABC$ whose inner angles are all 60°, then we see that no side of it can be longer than other ones (since for $\angle A$, if $\alpha > \beta$, then $1Bc > 1Ac$)

\[ \text{and} \]

if $1Bc > 1Ac$, then $\alpha > \beta$.

This means that all the sides of the triangle has the same length, i.e. the triangle is equilateral.

Similarly, if all the sides of a triangle are the same, we have that all the angles must be same, say $\alpha$. Since the sum of the inner angles of a triangle is 180° and all its angles are $\alpha$, $3\alpha = 180°$ & $\alpha = 60°$.

Which means if one of the statements is true, then the other statement is true too.

On the other hand, if a triangle is not equilateral, then it has one side, say $AC$ which is shorter than at least one of the other sides, say $1AC < 1AB$. Then $\gamma > \alpha$ and therefore they both cannot be 60°. Similarly we can show that if not all the angles are 60°, then the triangle cannot be equilateral.
Which means that whenever one of the statements is false, the other must be too.

Thus, showing a triangle is equilateral and all its inner angles are 60° are the same things. In this manner, an "iff" statement gives us equivalence of statements.

We can see this also on the truth table.

If we have the statements \(P\) and \(Q\), then:

\[
\begin{array}{|c|c|c|c|c|}
\hline
P & Q & P \Rightarrow Q & Q \Rightarrow P & (P \Rightarrow Q) \land (Q \Rightarrow P) \\
\hline
T & T & T & T & T \\
T & F & F & T & F \\
F & T & T & F & F \\
F & F & T & T & T \\
\hline
\end{array}
\]

ex: If \(x \cdot y = 0\) then \(x = 0\) or \(y = 0\), and conversely.

We can see this as

\[P \iff (Q \lor R)\]

where \(P\): \(x \cdot y = 0\), \(Q\): \(x = 0\), \(R\): \(y = 0\).

(A note: this "iff" statement is true in integral domains and won't hold in spaces like "modular spaces", "the ring of matrices", but this is beyond the scope of this course.)

2.5 Truth tables for statements:

Recall that when we have statements \(P\) and \(Q\), if we want to talk about the statement "either \(P\) or \(Q\)". Now, since we know the "not" notation we can write this statement in terms of "\(\land\)" "\(\lor\)" and "\(\neg\)".
When we say "Either P or Q" what we mean is, "P or Q but not both" i.e. \( (P \lor Q) \land \neg (P \land Q) \).

The question, now, is whether this expression meets our expectation in terms of the truth values. We need this statement to be true exactly when P or Q is true but not when they are both true. Let's check it in the truth table.

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<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>( (P \lor Q) )</th>
<th>( (P \land Q) )</th>
<th>( \neg (P \land Q) )</th>
<th>( (P \lor Q) \land \neg (P \land Q) )</th>
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If we look at the truth table, we see that the statement \( (P \lor Q) \land \neg (P \land Q) \) is true (rows 2 & 3) when \( P \) is true & \( Q \) is false and when \( P \) is false and \( Q \) is true (row 2) but it is false they are both true and when they are both false (row 4).

Now, let's see another example.

Ex: Given \( x, y \in \mathbb{R} \), the product \( xy = 0 \) if \& only if \( x = 0 \) or \( y = 0 \).

Write this statement in symbolic logic notation (i.e., in term of letters, and symbols "\( \land, \lor, \neg, \Rightarrow \)".
(let \( P : x \cdot y = 0 \) \( Q : x = 0 \) \( R : y = 0 \) \), then the statement is \( P \iff (Q \lor R) \).

Now let's see when this statement is true using truth table.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>QVR</th>
<th>P \iff (QVR)</th>
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We see that \( P \iff (Q \lor R) \) is true when \( P \) is true and at least one of \( Q \) or \( R \) is true and when \( P \) is false and \( Q \) and \( R \) are both false.

2.6: **Logical Equivalence**

In class, we saw that a statement \( P \iff Q \) is not the same as its converse \( Q \iff P \) by simply looking at a simple example (If it rains \( \iff \) roads get wet) and as well as their truth tables.
But then we may ask the question: if \( P \Rightarrow Q \) is not logically the same as \( Q \Rightarrow P \), what is? Or, is there such a statement which is logically same as \( P \Rightarrow Q \)? The answer is: Yes! This, indeed, is going to make our job in proving certain theorems much simpler.

\[ \text{Ex: let } x \in \mathbb{Z}. \text{ If } x^2 - 6x + 5 \text{ is even, then } x \text{ is odd.} \]

First call \( P: x^2 - 6x + 5 \text{ is even,} \]
\[ Q: x \text{ is odd.} \]
Then our statement is: \( P \Rightarrow Q \).

We saw that when \( P \) is false, this statement is automatically true. So, to prove that this statement is actually always true, we assume \( P \) is true and try to show \( Q \) is true as well. But in this particular situation we have a problem. Because if we assume \( P \) is correct, we need to assume that \( x^2 - 6x + 5 \text{ is even, which means that for every possible even value } x^2 - 6x + 5 \text{ may take, we need to show } x \text{ is odd. Which, of course, is impossible, since there are infinitely many possible even values } x^2 - 6x + 5 \text{ can take.} \]
So we need a better plan...

To prove this statement, if we pay a closer look, all we have to do is to realize that if \( x \) were even, then \( x^2 \) would be even 6\( x \) would be even \( x^2 - 6x \) would be even and thus \( x^2 - 6x + 5 \) would be odd.

But it is given that \( x^2 - 6x + 5 \) is even, and so, there is no way \( x \) could be even and thus, \( x \) should be odd. We can see, this argument is not only valid, but also very simple. If we pay close attention, what we did is actually to prove \( x \) is even \( \Rightarrow \) \( x^2 - 6x + 5 \) is odd, which is \( \neg\alpha \Rightarrow \neg\beta \).

Now, the question is: Is this always the case that \( P \Rightarrow \alpha \) is logically same as \( \neg\alpha \Rightarrow \neg\beta \)? The answer is: Yes! This even has a name: \( \neg\alpha \Rightarrow \neg\beta \) is the contrapositive of \( P \Rightarrow \alpha \). The easiest way to see that these statements are logically the same, we can look at their truth table. If they are true and false in exactly the same cases.
then we can say that they are logically equivalent and thus, instead of showing \( P \supset Q \) is true, it is necessary and sufficient for \( \neg Q \supset \neg P \) to be true. Namely

<table>
<thead>
<tr>
<th></th>
<th>( P )</th>
<th>( \neg P )</th>
<th>( \neg Q )</th>
<th>( P \supset \neg Q )</th>
<th>( \neg Q \supset \neg P )</th>
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</thead>
<tbody>
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<td>T</td>
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</table>

Here we see that their truth tables are the same.

**Ex:** (DeMorgan's laws)

- \( \neg (P \land Q) = \neg P \lor \neg Q \)
- \( \neg (P \lor Q) = \neg P \land \neg Q \)

let's show the first one (second one is also done similarly)

<table>
<thead>
<tr>
<th></th>
<th>( P )</th>
<th>( \neg P )</th>
<th>( \neg Q )</th>
<th>( \neg (P \land Q) )</th>
<th>( \neg P \lor \neg Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</table>

these columns are the same.
Moreover, we have more logical equivalences those can easily be verified using truth tables.

- \( P \Rightarrow Q = (\neg P) \Rightarrow (\neg Q) \)  contrapositive law
- \( \neg (P \land Q) = \neg P \lor \neg Q \) \hspace{1cm} \text{DeMorgan's laws}
- \( \neg (P \lor Q) = \neg P \land \neg Q \)  
- \( P \land Q = Q \land P \)  \hspace{1cm} \text{Commutative law}
- \( P \lor Q = Q \lor P \)  
- \( P \land (Q \lor R) = (P \land Q) \lor (P \land R) \) \hspace{1cm} \text{Distributive laws}
- \( P \lor (Q \land R) = (P \lor Q) \land (P \lor R) \)  
- \( P \land (Q \land R) = (P \land Q) \land R \)  \hspace{1cm} \text{Associative laws}

We can also verify

**ex:**  \( P \Rightarrow Q = (\neg P) \lor Q \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
P & Q & \neg P & P \Rightarrow Q & \neg P \lor Q \\
\hline
T & T & F & T & T \\
T & F & F & F & F \\
F & T & T & T & T \\
F & F & T & T & T \\
\hline
\end{array}
\]

these two columns are the same, meaning that \( P \Rightarrow Q \) is logically equivalent to \( \neg P \lor Q \).
2.7 Quantifiers:

Although "\(\land, \lor, \land, \Rightarrow, \Leftrightarrow\)" are useful in converting sentences into expressions in symbolic logic, they are not enough for mathematical sentences. For this, we introduce the notations "\(\forall\)", which reads as "for all" or "for every" and "\(\exists\)" which reads as "there exists" or "there is a". 

ex: There exists a subset \(X\) of \(\mathbb{N}\) s.t \(1 \times 1 = 5\) can be written as 
\[\exists X, (x \in \mathbb{N}) \land (1 \times 1 = 5)\] or 
\[\exists X \subseteq \mathbb{N}, 1 \times 1 = 5\]

or 
\[\exists X \in \mathcal{P}(\mathbb{N}), 1 \times 1 = 5.\]

These symbols \(\forall\) & \(\exists\) are called quantifiers. \(\forall\) is called universal quantifier and \(\exists\) is called existential quantifier.

ex: "For every real number \(x\), there is a real number \(y\) for which \(y^3 = x\).", can be written as 
\[\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^3 = x.\]

Which is a true statement (with \(y = \sqrt[3]{x}\), since cubic root is defined for every \(x \in \mathbb{R}\)).

Note: We have to be careful with the order of these quantifiers. The statement "\(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^3 = x\)" is not the same as "\(\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y^3 = x\)" although
they have exactly the same quantifiers.

If we pay attention, the second statement says
"There exists a real number \( y \), such that
for every real number \( x \), \( y^3 = x \)."

which means
"There exists a real number \( y \), s.t. \( y^3 = x \)
for every given \( x \)."

We clearly see that this statement is false.
So the order of quantifiers is crucial.

\[ \text{ex: let } (x_n) \text{ be a sequence, then we say that} \]
\[ (x_n) \text{ converges iff: there exists } L \in \mathbb{R} \text{ s.t. } (x_n) \text{ gets}
\]
\[ \text{arbitrarily close to } L \text{ as } n \to \infty. \]

: There exists \( L \in \mathbb{R} \) s.t. for every \( \varepsilon > 0 \),
all the \((x_n)'s\) for \( n \) sufficiently large, \( |x_n - L| < \varepsilon \).

: There exists \( L \in \mathbb{R} \) s.t. for every \( \varepsilon > 0 \), there
is \( N_0 \in \mathbb{N} \), s.t. for every \( n > N_0 \), \( |x_n - L| < \varepsilon \)

: \( \exists L \in \mathbb{R}, \forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall n > N_0, |x_n - L| < \varepsilon. \)

This may look like a complicated example, but all
we did was to take our "intuitive defn" for a sequence
to converge, and expressed every statement rigorously.

**Remark:** If we have an open sentence or a statement
that, \( P \), that contains a variable \( x \), we can write
\( P \) as \( P(x) \) and given a set \( S \), a quantified statement
of the form \( \forall x \in S, P(x) \), is true if \( P(x) \) is true for every \( x \in S \). If there is at least one element \( y \in S \) such that \( P(y) \) is false, then \( \forall x \in S, P(x) \) is false.

\[ \forall x \in (-1, 1), \ x^2 < |x| \text{. This statement is false since if we take } x = 0, \ 0^2 = 0, \text{ not } 0^2 < 0, \text{ although for every } x \in (-1, 1) \text{ other than } x = 0, \text{ this statement is true. (i.e. the statement } \forall x \in (-1, 1), x^2 < |x| \text{ is true.)} \]

**Recall:** Even when \( P \) & \( Q \) are open sentences, "If \( P(x) \), then \( Q(x) \)" is still a statement which can be false if \( Q(x) \) is false whenever \( P(x) \) is true.

So to prove such a statement is true, we assume \( P(x) \) to be true and try to show \( Q(x) \) follows.

**Ex:** If \( \lim_{n \to \infty} x_n = L \neq 0 \), then \( \lim_{n \to \infty} \frac{1}{x_n} = \frac{1}{L} \).

To prove this, we first assume that \( (x_n) \) is a convergent sequence, converging to a nonzero limit \( L \), then we try to show that \( \lim_{n \to \infty} \frac{1}{x_n} = \frac{1}{L} \) necessarily follows.

If we take a divergent sequence, or a sequence that converges to 0, then recall that our conditional statement is automatically true.

(Those who are curious about the proof of this statement can mail me and I can send the detailed proof.)
In logical equivalence chapter, we saw that if we negate the statement \( (P \land Q) \), i.e., if we look at \( \lnot (P \land Q) \), we get \( (\lnot P) \lor (\lnot Q) \), which was one of the DeMorgan's laws. Now, of course, the immediate question to ask is "what happens when we negate a statement with quantifiers." For example, if we want to show that a given sequence \((x_n)\) does not converge, what should we do? How can we negate the statement "\( \exists \epsilon > 0, \\forall n \in \mathbb{N}, \forall n \geq N, |x_n - L| < \epsilon \)"?

Let's start with simpler examples:

ex: Negate the statement "Square of every real number is nonnegative", i.e., express when it is not the case that square of every number is nonnegative.

We see that if it is not the case that square of every number is nonnegative, then there must be at least one real number whose square is negative. Now let's see this in symbolic logic notation. Our statement was "Square of every real number is nonnegative" which can be written as

"\( \forall x \in \mathbb{R}, x^2 \geq 0 \)". We said that its negation should be "there exists at least one real number whose square is negative" i.e.

"\( \exists x \in \mathbb{R}, x^2 < 0 \)"
In fact, for a general statement 
\( \forall x \in S, \, P(x) \), its negation is 
\( \neg (\forall x \in S, \, P(x)) = \exists x \in S, \, \neg P(x) \).

If we apply a similar argument to the existential quantifiers, we see
\( \neg (\exists x \in S, \, P(x)) = \forall x \in S, \, (\neg P(x)) \).

Let's see another statement with more quantifiers.

ex: Negate the statement
\( S: \) For every real number \( x \), there is a real number \( y \) for which \( y^3 = x \).

First, write \( S \) in symbolic logic form.

\( S: \forall x \in \mathbb{R}, \, \exists y \in \mathbb{R}, \, y^3 = x. \)

\[ \downarrow \]

\( \neg S = \neg (\forall x \in \mathbb{R}, \, \exists y \in \mathbb{R}, \, y^3 = x) \)

\[ = \exists x \in \mathbb{R}, \, \forall y \in \mathbb{R}, \, y^3 \neq x \]

which means, the negation of the statement \( S \) is
\( \neg S: \) There exist an \( x \in \mathbb{R} \), such that for all \( y \in \mathbb{R} \), \( y^3 \neq x \).

We mentioned in the class and also in these notes, that 
\( P \Rightarrow Q \) is logically equivalent to \( \neg (P \land \neg Q) \). Then we see that if we need to negate an implication \( P \Rightarrow Q \),
we get 
\( \neg (P \Rightarrow Q) = \neg (\neg P \land Q) = \neg \neg P \land \neg Q \) (De Morgan's law)
\[ = P \land \neg Q. \]
ex: Negate \( R: \) If \( a \) is odd, then \( a^2 \) is odd, for a given \( a \in \mathbb{R} \).

This statement can be written as

\[
P \Rightarrow Q \quad \text{for} \quad P: \ a \text{ is odd} \quad Q: \ a^2 \text{ is odd}.
\]

Then \( nR: n(P \Rightarrow Q) = P \land \overline{Q} \)

\[
\begin{align*}
= a \text{ is odd and } a^2 \text{ is not odd} \\
= a \text{ is odd and } a^2 \text{ is even.}
\end{align*}
\]

ex: Negate \( R: \) If \( x \) is odd then \( x^2 \) is odd.

Now, there is a little difference here. In the previous example, \( a \) was a fixed given number even though we didn't know what it was. But now we have a variable \( x \), and if you remember, when we have open sentences in an "if-then" statement, i.e., our statement is of the form \( P(x) \Rightarrow Q(x) \), then it has to hold for every \( x \) given in its domain. This means, \( R \) can be rewritten as

\[
R: \forall x, (x \text{ is odd}) \Rightarrow (x^2 \text{ is odd})
\]

\[
R: \forall x, P(x) \Rightarrow Q(x) \quad \text{for} \quad P(x): x \text{ is odd} \quad Q(x): x^2 \text{ is odd}.
\]

Hence \( nR: n(\forall x, P(x) \Rightarrow Q(x)) \)

\[
= \exists x, n(P(x) \Rightarrow Q(x)) = \exists x, P(x) \land \neg Q(x).
\]

which reads as \( nR: \) "There is an \( x \) s.t \( x \) is odd and \( x^2 \) is even."

Note: Suppose \( S \) is a set and \( Q(x) \) is a statement about \( x \) for each \( x \in S \). The following statements mean the same thing: "\( \forall x \in S, Q(x) \)" and "(\( x \in S \) \( \Rightarrow \) \( Q(x) \)".
We see that the first one reads as: "for every $x$ in $S$, we have $Q(x)$" which means "whenever $x$ is in $S$, we have $Q(x)$" which is the same as "$x \in S \Rightarrow Q(x)$".

This observation is going to be particularly important when we negate conditional statements with open sentences as in the previous example.

Now, let's see a more complicated example.

Ex: Write when a sequence $(x_n)$ does not converge.

We saw that

$$(x_n \text{ converges}) \iff \exists L \in \mathbb{R}, \forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall n \geq N_0, |x_n - L| < \varepsilon.$$  

This means that $(x_n)$ does not converge, when right hand side of "$\iff$" is "not the case" i.e. when

$$n(\exists L \in \mathbb{R}, \forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall n \geq N_0, |x_n - L| \geq \varepsilon)$$

$$= \forall L \in \mathbb{R}, \forall \varepsilon > 0, \forall N_0 \in \mathbb{N}, \forall n \geq N_0, |x_n - L| \geq \varepsilon.$$  

which reads as: "for any given real number $L$, there is an $\varepsilon > 0$, such that for all $N_0$ in $\mathbb{N}$, there is an $n \geq No$ such that the distance between $(x_n)$ & $L$ is greater than or equal to $\varepsilon$. "