Math 215 Section 201

Lecture notes 1

Introduction:

First, let’s recall some definitions.

Given an $n$-times differentiable function $y$ that depends on one variable, say $t$, the equation

$$F(t, y, y', y'', ..., y^{(n)}) = 0$$

where $F$ is a multivariable function is called an ordinary differential equation.

If $y$ is a multivariable function and the partial derivatives exist in the equation, then the equation is called a partial differential equation.

The highest order of derivative that appears
in the equation is called the order of a differential equation.

Example: \( y''' + 2e^t y'' + yy' = t^q \) is a third order differential equation, where \( y = y(t) \).

We call a differential equation

\[ L(y) = F(t, y, y', \ldots, y^{(n)}) = 0 \]

a linear equation if \( F \) is a linear func of the variables \( y, y', \ldots, y^{(n)} \). What this means is that if we replace \( y \) with \( y_1 + ty_2 \), and of course all the derivatives with them, then we get

\[ L(y_1 + ty_2) = L(y_1) + L(ty_2). \]

And if we replace \( y \) with \( cy \) where \( c \) is a constant, and the corresponding derivatives accordingly, we get
\[ L(cy) = c \cdot L(y). \]

An equation that is not linear is called a nonlinear equation.

**Ex:** \( y'' + 3ty' + e^ty = 0 \) is a linear equation since if we replace \( y \) with \( y_1 + ty_2 \), we get

\[
L(y_1 + ty_2) = (y_1 + ty_2)'' + 3t(y_1 + ty_2)' + e^t(y_1 + ty_2)
\]

by derivative of a sum is sum of derivatives and derivative distributes over sums.

\[
= (y_1'' + ty_2'') + 3t(y_1' + ty_2') + e^t(y_1 + ty_2)
\]

\[
= (y_1'' + 3ty_1' + e^ty_1) + (y_2'' + 3ty_2' + e^ty_2)
\]

\[
= L(y_1) + L(y_2).
\]

Similarly \( L(cy) = (cy)'' + 3t(cy)' + e^t(cy) \)

\[
= cy'' + c3ty' + ce^ty
\]

\[
= c(y'' + 3ty' + e^ty) = cL(y)
\]

where \( c \) is a constant.
Thus the eqn \( y'' + 3ty' + e^t y = 0 \) is a linear eqn.

Ex: The eqn \( y'' + y' = 0 \) is not linear since

\[
(y_1 + y_2)''(y_1 + y_2) + (y_1 + y_2)' = (y_1'' + y_2'')(y_1 + y_2) + (y_1' + y_2')
\]

\[
\nequiv (y_1'', y_1 + y_1') + (y_2'', y_2 + y_2')
\]

We see that the general linear ordinary differential eqn of order \( n \) is

\[
a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t).
\]

A solution of the \( n \)th order ord. diff'l eqn

\[
F(t, y, y', \ldots, y^{(n)}) = 0
\]

on the interval \( \alpha < t < \beta \) is a func \( \phi \), such that \( \phi, \phi', \ldots, \phi^{(n)} \)
exist & satisfy

\[
F(t, \phi(t), \phi'(t), \ldots, \phi^{(n)}(t)) = 0.
\]
Ex: Consider the eqn \( y'' + y = 0 \).

We see that this is a second order linear ord. diff' l eqn and the function 
\( y_1 = \cos t \) is a soln since 
\( y', \text{ } \& \text{ } y'' \) exist \& moreover \( y_1' = -\sin t \)
\& \( y_1'' = -\cos t \), which means
\( y_1'' + y_1 = -\cos t + \cos t = 0 \).

We can also check that \( y_2 = \sin t \) is also a soln.

**First Order Differential Equations**

Chapter 2.1: Linear Diff' l eqns, Integrating Factors

In this chapter we are going to learn
the method of integrating factors to solve a
general first order linear differential equations in
the standard form

\[(\ast) \quad \frac{dy}{dt} + p(t)y = g(t) \quad \text{where } p(t) \quad \text{&} \quad g(t) \quad \text{given func of the variable } t, \quad \text{or} \quad \text{of the form} \]

\[(\ast\ast) \quad p(t) \quad \frac{dy}{dt} + q(t)y = g(t), \quad \text{where} \]

\[p, q \quad \text{&} \quad g \quad \text{are given func. We can easily see that if } p(t) \to 0, \text{ we can write} \]

\[(\ast\ast) \quad \text{as an eqn of the form (\ast) by dividing the eqn by } p(t). \]

To understand the method, we need to recall the product rule of differentiation:

If we have two different func. \(x \quad \text{&} \quad y, \)

then \((xy)' = x'y + xy' \quad \text{---- (\ast\ast)\)}

But if we look at the eqn (\ast),

\[y' + p(t)y = g(t)\]
we see that the left hand side (LHS) of this eqn resembles \((***)\), but not quite the same. We see however that if we multiply the eqn \((*)\) by \(\mu(t)\), we get

\[ \mu(t) y' + \mu(t)p(t)y = \mu(t)g(t). \quad (4) \]

This tells me that if we have

\[ \mu(t)p(t) = \mu'(t) \]

then the left hand side of \((4)\) becomes a full derivative, \(\mu y'\).

Now, first let's find out what \(\mu\) should be. If we assume temporarily that \(\mu(t)\) is positive, dividing \(\mu(t)p(t) = \mu'(t)\) by \(\mu(t)\), we get

\[ p(t) = \frac{\mu'(t)}{\mu(t)}. \]

Nice thing about this eqn is that we see that the right hand side (RHS) is a full derivative, namely

\[ \frac{\mu'(t)}{\mu(t)} = (\ln(\mu(t)))' = p(t). \]

Thus, integrating
both sides, we get

$$\ln(1/\mu(t)) = \int p(t) dt + k$$

i.e. $$\mu(t) = \exp(\int p(t) dt)$$

But then, with this $$\mu$$, the eqn (4) becomes

$$\mu(y) = \mu(t)g(t) \quad \text{& hence}$$

by integrating both sides we get

$$\mu(t)y(t) = \int \mu(t)g(t) dt + C \quad \text{& hence}$$

$$y(t) = \frac{1}{\mu(t)} \left( \int_{t_0}^t \mu(s)g(s) ds + C \right).$$

where $$t_0$$ is some convenient lower limit of integration.

This may look a little intimidating, but in reality, the method is quite straightforward & if we remember where the formulas come from, solving these eqns will be easy.
Now, let’s see it in an example.

Ex: Solve the initial value problem

\[
\begin{align*}
    ty' + 2y &= 4t^2 \\
    y(1) &= 2
\end{align*}
\]

To solve this eqn, let’s put it in the standard form \((x)\) first, by dividing it by \(t\). Thus, we get

\(5) \quad y' + \frac{2}{t} y = 4t.\) Here, we see that in general form, \(p(t) = \frac{2}{t}\) & \(g(t) = 4t.\)

Now, first thing we need to do is to compute the integrating factor \(\mu(t)\):

\[
\mu(t) = \exp\left(\int \frac{2}{t} \, dt\right) = e^{\ln(t^2)} = t^2
\]

Then, multiplying the eqn \((5)\) by \(\mu(t) = t^2\), we get

\[t^2 y' + 2t y = 4t^3. \quad --- (6)\]
(But then, from the previous arguments we had, we see that the LHS of (6) must be the derivative of \( \mu(t)y = t^2y \), and indeed \((t^2y)' = t^2y' + 2ty\).)

Hence we get

\[
(t^2y)' = 4t^3 \quad \text{which implies}
\]

\[
t^2y = \int 4t^3 \, dt = t^4 + C, \quad \text{where} \ C \ \text{is an arbitrary constant. Thus} \ y = \frac{1}{t^2}(t^4 + C)
\]

i.e. \( y(t) = t^2 + \frac{C}{t^2} \).

Although \( y(t) \) satisfy the eqn itself, it also has to satisfy the boundary condition

\( y(1) = 2 \). This means that

\[
y(1) = 1^2 + \frac{C}{1^2} = 1 + C = 2.
\]

Thus \( C = 1 \).

Therefore the solution is \( y(t) = t^2 + \frac{1}{t^2}, t > 0 \).
Remark: We see that this solution fails to exist for all times and because of the infinite discontinuity in \( p(t) \) at \( t=0 \), which restricts the solution to the interval \( t>0 \) (or \( t<0 \), but since the initial data is given at \( t=0 \), we take the interval \( t>0 \)).

Ex: Solve the initial value problem

\[
\begin{cases}
2y' + ty = 2 \\
y(0) = 1
\end{cases}
\]

In this question, to be able to apply the method we have seen above, we need to write the eqn in the standard form first, by dividing the eqn by 2 & thus making the coefficient of \( y' \) 1. Then we have the

\[
\frac{2y' + ty}{2} = \frac{2}{2} \Rightarrow y' + \frac{t}{2}y = 1
\]
\[
\begin{bmatrix}
y' + \frac{1}{2} y = 1 \\
y(0) = 1
\end{bmatrix}
\]

Thus, we see that \( p(t) = \frac{1}{2} \) and \( q(t) = 1 \), as given in the formula.

Then we first find the integrating factor
\[
m(t) = \exp\left(\int \frac{1}{2} \, dt\right) = e^{t/4}.
\]

Then, multiplying the eqn in standard form by \( m(t) = e^{t/4} \) we get
\[
e^{t/4} y' + \left( e^{t/4} \cdot \frac{1}{2} \right)y = e^{t/4}
\]
\[\overline{\left( e^{t/4} y \right)'}\]

\[
\Rightarrow \quad (e^{t/4} y)' = e^{t/4}. 
\]
Thus, integrating both sides, we get, by fundamental theorem of calculus,
\[
e^{t/4} y = \int e^{t/4} \, dt + c, \quad \text{where} \ c_1 \ \text{is a constant.}
\]
arbitrary constant. Since the initial condition is given at $t=0$, and that the function $e^{t^2/4}$ doesn't have an explicit antiderivative, we can write the eqn above as

$$e^{t^2/4} y = \int_0^t e^{s^2/4} \, ds + C$$

again, $C$ is an arbitrary constant. Then we see

$$y(t) = e^{-t^2/4} \int_0^t e^{s^2/4} \, ds + C e^{-t^2/4},$$

and since $y(0)=1$, we get

$$1 = y(0) = e^0 \int_0^0 e^{-s^2/4} \, ds + C e^0,$$

$$1 = 0 \text{ (integral of a continuous func from 0 to 0)}$$

$$\Rightarrow 1 = C.$$

Therefore

$$y(t) = e^{-t^2/4} \int_0^t e^{s^2/4} \, ds + e^{-t^2/4}.$$
Separable eqns: We have seen that a general 1st order differential eqn can be written as (in the independent variable x)

\[ M(x, y) + N(x, y) \frac{dy}{dx} = 0. \quad \ldots \quad (5) \]

In this section, we are going to take a look at a specific case of such eqns where \( M \) doesn’t depend on \( y \) & \( N \) doesn’t depend on \( x \), i.e., \( M(x, y) = M(x) \) & \( N(x, y) = N(y) \). Then the eqn (5) turns into

\[ (6) \quad M(x) + N(y) \frac{dy}{dx} = 0. \]

Such an equation is called separable, because if it is written in differential form

\[ M(x) \, dx + N(y) \, dy = 0, \]

we can separate the \( x \) & \( y \) variable terms into opposite sides of the eqn.
To solve the eqns of the form (6), assume that \( H_1(x) \) & \( H_2(y) \) are antiderivatives of \( M(x) \) & \( N(y) \) respectively, that is

\[
M(x) = \frac{d}{dx} H_1(x) \quad \text{&} \quad N(y) = \frac{d}{dy} H_2(y).
\]

Then we see that the eqn (6) becomes

\[
\frac{d}{dx} H_1(x) + \frac{d}{dy} H_2(y) \frac{dy}{dx} = 0.
\]

This is just \( \frac{d}{dx} H_2(y) \)

using chain rule

\[\Rightarrow \frac{d}{dx} H_1(x) + \frac{d}{dx} \left( H_2(y) \right) = 0. \]

\[\Rightarrow \frac{d}{dx} \left( H_1(x) + H_2(y) \right) = 0, \text{ which means } \]

\[H_1(x) + H_2(y) = C \quad \text{where } c \text{ is a constant.} \]

Moreover, if the initial condition \( y(x_0) = y_0 \),
is given, we see

\[ c = H_1(x_0) + H_2(y_0) \]

Then, putting this \( c \) in the eqn above, and noticing that

\[ H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s) \, ds \]

\[ H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s) \, ds \] (by Fundamental theorem of Calc.)

we get

\[ \int_{x_0}^{x} M(s) \, ds + \int_{y_0}^{y} N(s) \, ds = 0. \]

Heuristically what this tells us is that when we write eqn (6) in the form

\[ M(x) \, dx + N(y) \, dy = 0, \]

we can integrate both sides to get

\[ \int M(x) \, dx + \int N(y) \, dy = 0. \]

Now, let's see this in on example.
Ex: Solve the initial value problem

\[
\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} , \quad y(0) = -1.
\]

\& determine the interval in which the soln exists.

We see that this eqn can be written as

\[
2(y-1) \frac{dy}{dx} = (3x^2 + 4x + 2) \quad \text{or as}
\]

\[
2(y-1)dy = (3x^2 + 4x + 2)dx. \quad \text{Then, by the arguments above, integrating both sides we get (LHS in y, RHS in x)}
\]

\[
y^2 - 2y = x^3 + 2x^2 + 2x + c \quad \text{where c is an arbitrary constant.}
\]

Then, checking the initial condition, we get

\[
y^2(0) - 2y(0) = 0^3 + 20^2 + 200 + c \quad \text{and since}
\]
\[ y(0) = -1, \text{ we get} \]
\[ (-1)^2 - 2(-1) = c \implies c = 3. \]

Therefore, the solution satisfies the equation
\[ y^2 - 2y = x^3 + 2x^2 + 2x + 3 \]
& completing the LHS into a complete square, we get
\[ y^2 - 2y + 1 = (y - 1)^2 = x^3 + 2x^2 + 2x + 4 \]
\[ \implies y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4} \]
& since \( y(0) = -1 \), we see that we need to take the "-" sign i.e.
\[ y(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \]

To find the interval on which the solution is valid, we need to find the interval on which the expression inside square-root is positive. The only real root of the
expression \( x^3 + 2x^2 + 2x + 4 \) is \( x = -2 \) & for \( x > -2 \), \( y(x) \) is well defined. Therefore the soln is \( y = 1 - \sqrt[3]{x^3 + 2x^2 + 2x + 4} \) on \( x > -2 \).

You can see the integral curves of the eqn \( y' = \frac{3x^2 + x + 2}{2(y - 1)} \).
Ex: Solve the separable differential eqn

$$\frac{dy}{dx} = \frac{4x-x^3}{4+ y^3}$$

and draw graphs of several integral curves.

Find the soln that passes through the point $(0,1)$ & find its interval of existence.

First let's write the eqn in differential form as

$$(4+ y^3)dy = (4x-x^3)dx.$$ 

Then, integrating both sides, we get

$$4y + \frac{y^4}{4} = 2x^2 - \frac{x^4}{4} + c,$$

where $c \in \mathbb{R}$.

Now, if we multiply both sides by 4 & collecting $x$ & $y$ terms together, we get

$$y^4 + 16y + x^4 - 8x^2 = c.$$
Unfortunately there is no easy way to solve a 4th order polynomial (since any polynomial of order greater than 3 doesn't have radical soln - formula like quadratic formula). So we leave the soln in implicit form. If we check the initial condition \( y(0) = 1 \), we see

\[ y(0) + 16y(0) + 0 - 0 = c \Rightarrow 1 + 16 = c \Rightarrow c = 17. \]

Thus, the soln that passes through \((0,1)\) is

\[ y' + 16y + x^4 - 8x^2 = 17. \]

Now, if we look at the slope field & the integral curves of this eqn, we see the curve that passes thru \((0,1)\) fails to be a finc when tangent line is vertical i.e. when

\[ \frac{dy}{dx} = \frac{4x - x^3}{y^2}, \]

goes to infinity. That is,
when \( 4 + y^3 = 0 \Rightarrow y = -\sqrt[3]{4} \) & the \( x \) pts that correspond to this value of \( y \) are \( x = \pm 3.3488 \).

Chapter 2.3: Modeling with 1st order diff. eqs.

Differential eqns, ordinary or partial, are one of the main tools used to understand physical phenomena. The process of expressing a physical, biological, chemical etc. phenomena
in mathematical terms is called mathematical modeling. There are 3 main steps in modeling.

**Step 1:** Construction of the model

**Step 2:** Analysis of the model where we try to solve or try to understand the differential eqns in the model.

**Step 3:** Comparison with Experiment or observation where we look at the soln & compare it to experimental or observational data.

Ex: At time $t=0$ a tank contains $Q_0$ lb of salt water dissolved in 100 gal of water. Assume that water containing $\frac{1}{2}$ lb of salt per gallon is entering the
tank at a rate of \( r \) gal/min, and that the well-stirred is draining from the tank at the same rate. Set up the initial value problem that describes the flow process. Find the amount of salt \( Q(t) \) in the tank at any given time, also find the limiting amount \( Q_L \) that is present after a long time. If \( r = 3 \) & \( Q_0 = 2Q_L \), find the time \( T \) after which the salt level is within \( \%2 \) of \( Q_L \). Also find the flow rate that is required if the value of \( T \) is not to exceed 45 min.

\[ r \text{ gal/min, } \frac{1}{4} \text{ lb/gal} \]
To set up the initial value problem all we need to observe that the change in salt levels is due to the salt flowing in & flowing out & thus the rate of change of the salt amount is

$$\frac{dQ}{dt} = \text{(rate of salt that goes in)} - \text{(rate of salt that goes out)}$$

Since we know that the flow of the salt water pouring in is constant r gal/min & with concentration $\frac{1}{4}$ lb/gal, we see the rate of change of the salt amount that goes in is

$$r \cdot \frac{1}{4} = \frac{r}{4} \text{ lb/min}.$$  

Now, let's calculate the rate of change of amount of salt that goes out at any time t. For that we need to do a similar argument as above & we will calculate
Rate out = (Concentration at time t) \cdot (flow rate)
= (Concentration at time t) \cdot r

But now, the question is, how can we calculate the concentration of salt in the tank at any given time? This is not too difficult, all we remember is that Concentration = \( \frac{\text{Amount}}{\text{Volume}} \)
and we know amount of salt at any given time is represented as \( Q(t) \). Volume of the tank is 100 gallons. Thus, concentration at any given time is \( \frac{Q(t)}{100} \). Therefore the rate out is \( r \cdot \frac{Q(t)}{100} \). This gives us the differential equation

\[
\frac{dQ}{dt} = r - \frac{rQ}{100}, \quad \text{with } Q(0) = Q_0
\]
Observations: If we keep pouring in a salt mixture of concentration \( \frac{1}{4} \) lb/gal, we would expect the salt level in the tank to converge to the same concentration level, \( \frac{1}{4} \) lb/gal, i.e., amount of salt should converge to \( \frac{1}{4} \cdot \frac{100}{\text{Volume of tank}} = 25 \text{ lbs.} \)

We also see that at this limiting value, the amount of salt shouldn't change. This means that if we set \( \frac{dQ}{dt} = 0 \), we should get the limiting amount of salt, i.e.,

\[
0 = \frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100} \implies Q = 25 \text{ lb.}
\]

Now, let's solve this differential eqn

\[
\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}.
\]
We see that this is a first order linear ODE, written in the standard form. For this eqn, we can use the integrating factor, where \( p(t) = \frac{r}{100} \) & \( q(t) = \frac{r}{4} \).

Then \( \mu(t) = \exp \left( \int \frac{r}{100} \, dt \right) = e^{\frac{rt}{100}} \). Then multiplying the eqn (which is already in std. form) by the integrating factor, we get

\[
e^{\frac{rt}{100}} \frac{dQ}{dt} + \frac{r}{100} e^{\frac{rt}{100}} Q = \frac{r}{4} e^{\frac{rt}{100}}.
\]

\[
\left( e^{\frac{rt}{100}} Q(t) \right)^{'} \cdot \text{Then, by integrating, we get}
\]

\[
e^{\frac{rt}{100}} Q(t) = \int \left( \frac{r}{4} e^{\frac{rt}{100}} \right) \, dt
\]

\[
\Rightarrow e^{\frac{rt}{100}} Q(t) = 25 e^{\frac{rt}{100}} + C
\]

\[
\Rightarrow Q(t) = \left( \frac{25 e^{\frac{rt}{100}} + C}{e^{\frac{rt}{100}}} \right) = 25 + C e^{-\frac{rt}{100}}
\]
where \( c \in \mathbb{R} \). Then, given the initial condition \( Q(0) = q_o \), we see \( c = q_o - 25 \).

\[
Q(t) = 25 + (q_o - 25) e^{-r t/100}.
\]

Thus, we see that

\[
\lim_{t \to \infty} Q(t) = 25.
\]

Now, for the second part of the problem, let \( r = 3 \) & \( q_o = 2Q_L = 50 \). Then the eqn for \( Q \) becomes

\[
Q(t) = 25 + (50 - 25) e^{-3 \% t/100}
\]

\[
= 25 + 25 e^{-0.03 t}.
\]

Now we want this to be within 2\% of 25 i.e. we want the value \( Q(t) \) to be between \((24.5, 25.5)\). But, since \( q_o = 50 \), we expect to get \( Q(t) < 25.5 \).
If we substitute \( Q(t_0) = 25.5 \) in the eqn for \( Q(t) \), we get

\[
25.5 = 25 + 25 e^{-0.03 t_0}
\]

\[
\Rightarrow 50 = 25 e^{-0.03 t_0} \quad \Rightarrow \quad \frac{1}{50} = e^{-0.03 t_0}
\]

\[
\Rightarrow 50 = e^{0.03 t_0} \quad \Rightarrow \quad t_0 = \frac{\ln(50)}{0.03} \approx 130.4 \text{ mins.}
\]

If we want to calculate the rate \( r \) so that we get 2\% of \( Q_c \) in \( T = 45 \) mins, we need to plug the values in the eqn

\[
Q(t) = Qt + (Q_0 - Qt) e^{-rt/100}
\]

where \( Q(t) = 25.5 \), \( Q_0 = 50 \), \( t = 45 \). Then, we get

\[
25.5 = 25 + (25) e^{-0.45 r}
\]

\[
\Rightarrow r = \frac{100}{45} \ln 50 \approx 8.69 \text{ m/hr}.
\]
Ex: A body of constant mass $m$ is projected away from the earth in a direction perpendicular to the earth's surface with the initial velocity $v_0$. Assuming that there is no air resistance, but taking into account earth's gravitational field with distance, find the expression for the velocity during the ensuing motion. Also find the initial velocity that is required to lift the body to a given maximum altitude $A_{\text{max}}$ above the surface of the earth and find the least velocity for which the body will not return to the earth (which is called the escape velocity).
To find the differential eqn of the velocity, we are going to use the eqn of motion,

\[ m \cdot a = F. \]

To be able to do that, we need to find the forces acting on the object, which in this case, is the gravitational force.

Recall that the gravitational force

\[ W(x) = \frac{-m \cdot M \cdot G}{(x+R)^2} \]

where \( M \) is the mass of the Earth and \( G \) is the gravitational constant.
Thus, we also see that the numerator is constant, and we know at $x=0$, the force $W(0) = -mg$. Thus, we see that

$$-mg = -\frac{mGM}{(0+R)^2} \Rightarrow MG = gR^2, \text{ i.e.}$$

$$W(x) = -\frac{mgR^2}{(x+R)^2}.$$

Therefore, since gravitational force is the only force acting on the object, we get

$$ma = m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2} \text{ with the initial condition } v(0) = v_0. \text{ Now, we see that we have two independent variables } t \text{ and } x. \text{ But we can solve this problem using chain rule: }$$

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dt} \frac{1}{v}.$$
Therefore, the eqn becomes
\[ m \frac{dv}{dt} = m \frac{dv}{dx} \sqrt{v} = -\frac{mgR^2}{(R+x)^2} \]

\[ \Rightarrow \frac{dv}{dx} \sqrt{v} = -\frac{gR^2}{(R+x)^2} \] (where we see that we write \( v \) as an eqn in \( x \))

We see that this eqn is separable.

If we write it in differential form, we get
\[ v \frac{dv}{dx} = -\frac{gR^2}{(R+x)^2} dx \]

& integrating both sides we get,
\[ \frac{v^2}{2} = \frac{gR^2}{(R+x)} + C. \]

Now, since at \( t=0 \), we have \( x=0 \), we get
\[ \frac{\sqrt{v_0}}{2} = \frac{gR^2}{R} + C \]
\[ \Rightarrow C = \frac{v_0}{2} - gR \]

\[ \Rightarrow v = \pm \sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}, \text{now, since} \]
the object is moving in \( +x \)-direction, we need to take the \( '+' \) sign above i.e.

\[
v = \sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}.
\]

To determine the maximum altitude \( A_{\text{max}} \), we need to set \( x = A_{\text{max}} \) & since it is the max distance, we set \( v = 0 \).

\[
A_{\text{max}} = \frac{v_0^2 R}{2gR - v_0^2}
\]

Thus, if we solve for \( v_0 \), we get

\[
v_0 = \sqrt{2gR \frac{A_{\text{max}}}{R+A_{\text{max}}}}.
\]

And we can find the escape velocity, \( v_e \), by sending \( A_{\text{max}} \) to \( \infty \). i.e

\[
v_e = \lim_{A_{\text{max}} \to \infty} v_0 = \sqrt{2gR} \approx 11.1 \text{ km/s}.
\]
Chapter 2.4: Differences Between Linear & Nonlinear eqns:

- Existence & uniqueness:

  Theorem 2.4.1: (Existence & uniqueness for first order linear eqns)

  If the func p & g are cts on an open interval $I: a < t < b$, containing the point $t=t_0$, then there exists a unique soln $y = f(t)$ to the eqn $y' + p(t)y = g(t)$ for each $t \in I$, and also satisfies the initial condition $y(t_0) = y_0$, where $y_0 \in \mathbb{R}$.

  (We have seen how to solve such eqns, where the soln depends on the integrability of $p(t)$ (to find $\mu(t)$, the integrating factor) & integrability of $(\mu, g)$. The conditions in the
Thus guarantees the existence of all these expressions.

Whereas for the nonlinear eqns, the theorem is fundamentally different.

**Theorem 2.4.2: Existence & Uniqueness**

For 1st order nonlinear differential equations.

Let the functions \( f \) & \( \partial f / \partial y \) be continuous in some rectangle \( a < t < b \), \( c < y < d \) containing the point \((t_0, y_0)\). Then, in some interval \( t_0 - h < t < t_0 + h \) contained in \( a < t < b \), there is a unique solution \( y = \phi(t) \) of the initial value problem

\[
y' = f(t, y) \quad \text{and} \quad y(t_0) = y_0.
\]

- We see that if we apply this theorem to

  linear eqns \( y' = -p(t)y + g(t) \), we see
that \( f(t,y) = -p(t)y + g(t) \) & \( \frac{df}{dy}(t,y) = -p(t) \) being cts is equivalent to \( p \) & \( g \) being cts (continuous). But by the light of Thm 2.4.1, we don't need the special subinterval \((t_0-h, t_0+h)\).

\[ \begin{aligned}
\text{Ex: Consider the eqn} \\
\begin{bmatrix}
    ty' + 2y = 4t^2 \\
y(1) = 2 
\end{bmatrix}
\end{aligned} \]

We see that this is a 1st order linear eqn, which can be written in standard form as 
\( y' + \left(\frac{2}{t}\right)y = 4t \). So \( p(t) = \frac{2}{t} \) & \( g(t) = 4t \).

We see that \( g(t) \) is a cts fnce whereas \( p(t) \) is cts on \((0, \infty)\) & \((-\infty, 0)\). Then given the initial condition \( y(1) = 2 \), by Thm 2.4.1, we would expect to have a unique soln
for 0 < t. Indeed, we solved this eqn &
got \ y(t) = t^2 + \frac{1}{t^2} \ , \text{ which exists on } t > 0.

\textbf{Ex:} \text{ Consider the initial value problem}

\[ \frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \ , \quad y(0) = -1. \]

In light of Thm 2.4.2, we see that

\[ f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)} \quad \text{&} \quad \frac{\partial f(x,y)}{\partial y} = \frac{-\left(3x^2 + 4x + 2\right)}{2(y-1)^2}. \]

Both of these facts are csss everywhere except for when y = 1. Therefore, we can find a unique soln in a rectangle around (0, -1) (the initial condition).

Indeed, we solved this eqn & got

\[ y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \quad \text{which exists for} \quad x > -2. \]
Moreover, if we change the initial cond. to \( y(0) = 1 \) (recall that this is where \( f(x, y) \) \& \( \frac{\partial f}{\partial y}(x, y) \) is discontinuous).

Then, solving the eqn again with this initial data, we get

\[
y = 1 + \sqrt[3]{x^3 + 2x^2 + 2x}
\]

which is not unique (two solns, one for "+" one for ")

But this is not a problem since the Thm 2.4.1 didn't predict this soln to be unique when \( y = 1 \).

Ex: Consider the initial value problem

\[
y' = y^{1/3}, \quad y(0) = 0.
\]

In this question, we see

\[
f(t, y) = y^{1/3} \quad \text{which is a cts fnc but}
\]

\[
\frac{\partial f}{\partial y}(t, y) = \frac{1}{3} y^{-2/3} \quad \text{does not exist when } y = 0.
\]
This means that, with the initial value given, Thm 2.4.2 doesn't apply. Indeed, this is a separable eqn & we can solve it quite easily (without focusing on the initial data).

We can write the eqn in differential form as

$$y^{-\frac{1}{3}} \, dy = dt \quad \& \quad \text{integrating both sides we get}$$

$$\frac{3}{2} \, y^{\frac{2}{3}} = t + c \quad \Rightarrow \quad y = \left(\frac{2}{3} \, (t + c)\right)^{3/2}$$

& checking the initial condition \( y(0) = 0 \), we get \( c = 0 \) \( \Rightarrow \) \( y = \left(\frac{2}{3} \, t\right)^{3/2} \).

But here, we also see that \( y = 0 \), the constant 0 fnc is also a soln to this eqn & also satisfies the initial cond.
This means that the solution is not unique. Moreover, there are infinitely many solutions of the form
\[ y = \begin{cases} 
0 & t < t_0 \\
\left(\frac{2}{3} (t-t_0)^{3/2} & t \geq t_0 
\end{cases} 
\]
for any \( t_0 > 0 \).

Ex: Solve the initial value problem
\[ y' = y^2, \quad y(0) = 1 \] & determine the interval in which the solution exists.

First, we see that \( f(y) = y^2 \) & \( \frac{df}{dy} = 2y \) which are both continuous everywhere. So, we are in the setup of Thm 24.2 which means that there has to be a unique solution in a rectangle around \((0,1)\).
Now, let's solve this eqn. We see that this is a separable eqn & in differential form, it is

\[ y^{-2} \, dy = dt \]

& integrating both sides, we get

\[ -y^{-1} = t + c \quad \text{ i.e. } \quad y = \frac{-1}{t + c}. \]

Checking the initial cond. \( y(0) = 1 \), we get \( c = -1 \) i.e. \( y = \frac{-1}{t - 1} = \frac{1}{1-t} \).

This tells us that the soln goes to \( \infty \) as \( t \to 1 \). Thus, the soln exists in \(-\infty < t < 1\), which we couldn't see from the eqn itself & the conditions on Thm 2.4.2.

This tells us that when dealing with differential eqns, the linear eqns are much
nicer & more predictable than nonlinear
diff'1 eqns.

Chapter 2.5: Autonomous Differential Eqns

In this chapter we are going to consider the eqns of the form

\[ \frac{dy}{dt} = f(y), \] i.e. where the

eqn doesn't depend explicitly on the independent variable. There are two important examples of such eqns:

1) Exponential growth: let \( y = \phi(t) \) be the population of a given species at time \( t \). If we assume that the rate of change of the population \( y \) is proportional to the current value of \( y \), that is
\[ \frac{dy}{dx} = ry \] where the proportionality constant \( r \) is called rate of growth or decline.

If \( r > 0 \), then the population is growing & if \( r < 0 \), then the population is declining.

If we assume that \( r > 0 \), we can solve this eqn either by integrating factor or separation. Say, we use separability of the eqn & write it as

\[ \frac{dy}{y} = rdt \]

Then integrating both sides we get

\[ \ln|y| = rt + c \quad \text{c} \in \mathbb{R} \]

\[ y = Ce^{rt} \quad C \in \mathbb{R} \]

Then, if the initial value \( y(0) = y_0 \) is given, we get

\[ C = y_0 \]

\[ y = y_0 e^{rt} \]
Which says that the population grows exponentially. (We can see this as a model for bacterial growth with sufficient food and space, where every cell doubles in a certain time, i.e., \( r = \ln 2 \)).

2) Logistic growth: In real life, the exponential growth is practically impossible since the population also affects the amount of food and space available and affects the growth. To take this into account we can replace the proportionality constant with a function of the population, \( y \), i.e., we get

\[
\frac{dy}{dt} = h(y) \cdot y.
\]

One of the most observed versions of this
eqn is when \( h(y) = r > 0 \) when \( y \) is small i.e. the population grows almost exponentially & \( h(y) \) decreases as \( y \) increases & \( h(y) < 0 \) for \( y \) sufficiently large i.e. population declines if it is too big (especially for the environment to support it).

Simplest such \( h(y) \) is \( h(y) = r - ay \) where \( a > 0 \). Using this fnc. we get

\[
\frac{dy}{dt} = (r - ay)y.
\]

This eqn is called the logistic eqn & can also be written as

\[
\frac{dy}{dt} = r \left(1 - \frac{y}{k}\right)y \quad \text{where} \quad k = \frac{r}{a}.
\]

In this form, \( r \) is called the intrinsic growth rate. We see that this eqn has
two constant solns \( y = \phi_1(t) = 0 \) & \( y = \phi_2(t) = k \) which are called the equilibrium solns since they correspond to no change in population. These solns can also be found by locating the roots of \( f(y) \), i.e., \( \frac{dy}{dt} = f(y) = 0 \). The zeroes of \( f(y) \) are also called critical pts.

When we have this type of eqns, we can analyse the eqn & have some understanding about the soln even without solving the eqn explicitly. The way we do this is by understanding \( f(y) \) & how it changes with respect to \( (\text{w.r.t.}) \ y \) & thus understand \( \frac{dy}{dt} \) which tells us
where the function is increasing or decreasing. To see how this is done, let's go back to the logistic eqn.

$$y' = r \left(1 - \frac{y}{k}\right)y = f(y).$$

We see that the func $f$ is a downward facing parabola & since we are looking at the logistic eqn which models population growth we take $y > 0$ (meaning there are nonnegative number of members of the population).

Then we see

In this region $f > 0$ meaning $y > 0$ which implies $y$ is increasing.

In this region $f < 0$ meaning $y$ decreasing.

Equilibrium solns.
The y axis together with the arrows, is called the phase line.

Then if we plot \( y \), we see

We also see that the eqn is

\[
y' = f(y) \quad \text{& thus}
\]

\[
y'' = \frac{d}{dt} f(y) = \frac{df(y)}{dy} \cdot y' = \frac{df(y)}{dy} \cdot f(y)
\]

\[
\text{chain rule}
\]

\[
y' = f(y)
\]

\[
= r \left(1 - \frac{y}{k}\right)y \cdot \left(1 - \frac{2y}{k}\right)
\]
Thus we see that for \( y \in (0, \frac{\nu}{2}) \),
\( y'' > 0 \) and thus the soln is concave up
and for \( y \in (\frac{\nu}{2}, k) \) the solns are concave down, and for \( y > k \) the solns are again concave up (although they are decreasing).

We also see that all the nearby solns of the equilibrium soln \( y = k \)
converge to \( y = k \). Such equilibrium solns
where the nearby soln converge to them are called stable solns. The equilibrium solns for which the nearby solns diverge away from are called unstable solns.

Then we see \( y = k \) is a stable soln whereas
\( y = 0 \) is an unstable soln.
Now, if we solve the eqn
\[ y' = r \left( 1 - \frac{y}{k} \right) y , \]
we get
\[
\frac{dy}{dt} = r \left( 1 - \frac{y}{k} \right) y = \frac{dy}{(1 - \frac{y}{k})y} = r dt
\]

Now we are going to integrate both sides, but to do that we will use partial fraction expansion for LHS.

\[
\frac{1}{y(1 - \frac{y}{k})} = \frac{A}{y} + \frac{B}{(1 - \frac{y}{k})}
\]

\[= \frac{A - \frac{Ay}{k} + By}{y(1 - \frac{y}{k})} \]

\[= \frac{A}{k} + \frac{B}{1 - \frac{y}{k}}\]

\[\Rightarrow L = A + y \left( B - \frac{A}{k} \right) \Rightarrow A = 1 \text{ and } B - \frac{A}{k} = 0 \]

\[\Rightarrow B = \frac{1}{k} .\]
Thus, the eqn becomes
\[
\left( \frac{1}{y} + \frac{1}{1 - y/K} \right) \, dy = r \, dt
\]

& integrating both sides we get
\[
\ln |y| - \ln \left| 1 - \frac{y}{K} \right| = rt + c
\]

\[
\ln \left| \frac{y}{1 - \frac{y}{K}} \right| = rt + c \quad \text{CE \in \mathbb{R}, y \neq K.}
\]

\[
\frac{y}{1 - \frac{y}{K}} = Ce^{rt} \quad \text{& given } y(0) = y_0, \quad C \in \mathbb{R}.
\]

we see \[C = \frac{y_0}{1 - \frac{y_0}{K}}\] & solving this

eqn for \(y\), we get
\[
y = \frac{y_0 \cdot K}{y_0 + (K - y_0) \cdot e^{-rt}}
\]
Then we see that if \( y_0 > 0 \), then
\[
\lim_{t \to \infty} y(t) = \frac{y_0 \cdot K}{y_0} = K.
\]

**Ex:** This logistic model has been applied to halibut population in Pacific Ocean.

Let \( y(t) \) be the biomass measured in kilograms at time \( t \). Then the parameters have the estimated values \( r = 0.71/\text{year} \), \( K = 80.5 \times 10^6 \) kg.

Then if the initial biomass is \( y_0 = 0.25 \text{ kg} \), find the biomass 2 years later. Also find the time \( t \) for which \( y(t) = 0.75 \text{ kg} \).

**Solu:** We know the solu to the logistic eqn is
\[
y = \frac{y_0 \cdot K}{y_0 + (K - y_0) e^{-rt}}.
\]

Now, we can plug in the values \( y_0, K \) &
In this eqn & find $y(2)$

$$y(2) = \frac{0.25K \cdot K}{0.25K + (K - 0.25K) e^{-0.71x2}}$$

$$= \frac{0.25K^2}{0.25K + (0.75K) e^{-1.42}}$$

$$= \frac{K}{1 + 3e^{-1.42}} \approx 46.7 \times 10^6 \text{ kg}$$

& to solve for $x$ for which $y(x) = 0.75K$, we can use the eqn

$$\frac{y}{1 - \frac{y}{K}} = \frac{y_0}{1 - \frac{y_0}{K}} e^{rt}$$

the eqn that the soln satisfies (4) above

then we get

$$t = -\frac{1}{r} \ln\left(\frac{(y_0/K)(1 - \frac{y_0}{K})}{(y/K)(1 - \frac{y}{K})}\right)$$

& plugging in $y_0 = 0.25K$ & $y = 0.75K$
we get \[ t = -\frac{1}{0.71} \ln \left( \frac{(0.25)(0.25)}{(0.75)(0.75)} \right) = \frac{1}{0.71} \ln 9 \approx 3.095 \text{ years.} \]

**Ex: Logistic growth with threshold:**

Consider the eqn

\[ \frac{dy}{dt} = -r \left( 1 - \frac{y}{T} \right) \left( 1 - \frac{y}{K} \right) y \text{ where} \]

\[ r > 0,\ 0 < T < K,\ \text{(again} \ t > 0 \ \text{&} \ y > 0) \]

**Soln:** We see that this eqn has 3 equilibrium solns \( y = 0,\ y = T \ \text{&} \ y = K \) where

\[ y' = f(y) = -r \left( 1 - \frac{y}{T} \right) \left( 1 - \frac{y}{K} \right) y = 0 \text{.} \]

Then, if we graph \( f(y) \), we get

---

**Phase line of \( y \):**

- **in this region** \( y' > 0 \).
- **in this region** \( y' < 0 \).

---
Then if we draw the graph of $y$, we get

\[ y \]

\[ K \]

\[ T \]

\[ 0 \]

\[ y_1 \]

\[ y_2 \]

\[ y = \phi_2(t) = K \]

\[ y = \phi_2(t) = T \]

\[ y > \phi_1(t) = 0 \]

(Again, the curvature of the solns change at $y_1$ & $y_2$)

In this question, we see that $y = K$ & $y = 0$ are stable solns whereas $y = T$ is an unstable soln.

This level $T$ is called a threshold level since growth only happens when $y(t) > T$ & for $y(t) < T$, the solns decay to 0.
Exercise: Solve this differential eqn explicitly.

**Second Order Linear Diff’l Eqns**

Chapter 3.1: Homogeneous Diff’l eqns with Constant Coefficients:

We saw that a second order linear diff’l eqn has the form

\[ y'' + p(t)y' + q(t)y = g(t) \]

or

\[ P(t)y'' + Q(t)y' + R(t)y = G(t) \]

where \( P(t) \neq 0 \) & \( p(t), q(t), g(t), P(t), Q(t), R(t), G(t) \) are specific funcs that don’t depend on \( y \).

We call these eqns homogeneous
if \( g(t) \) (or \( G(t) \)) is zero. Otherwise the eqn is called nonhomogeneous.

In this section we are going to consider a very specific case of homogeneous 2nd order diff eqn, the eqn with constant coefficients, i.e eqns of the form

\[
ay'' + by' + cy = 0 \quad \text{with} \quad y(t_0) = y_0, \quad y'(t_0) = y_1.
\]

To solve these type of eqns we are going to use what we call the similarity solns. For that, we would like to consider solns whose second & first derivatives are "similar" to the fnc itself & if we plug in such a soln
into the eqn, the derivatives, hopefully, disappear & we are left with an eqn in terms of the func itself. This may sound abstract, but actually quite straightforward. The question, now, is: what is such func? We see that the answer is \( y(t) = e^{rt} \) (or constant solns which are also included in \( r < 0 \) case).

Now, if we plug in \( y(t) = e^{rt} \) into the eqn we get
\[
ay'' + by' + cy = ar^2e^{rt} + bre^{rt} + cte^{rt} = 0.
\]

\[
e^{rt}(ar^2 + br + c) = 0
\]

\[
\ne^{rt} \neq 0
\]

\[
\Rightarrow ar^2 + br + c = 0.
\]

This eqn is called the
characteristic eqn of the diff'1
eqn $ay'' + by' + cy = 0$.

Then, using the solns to the
characteristic eqn, we can find
solns to the diff'1 eqn. Since the
characteristic eqn is a quadratic eqn,
we have 3 cases

Case 1: $ar^2 + br + c = 0$ has 2 distinct
real solns $r = r_1$ & $r = r_2$.

Then by the arguments above,
we see that $y_1 = e^{r_1t}$ & $y_2 = e^{r_2t}$
are solns to the eqn.

Then, since the differential eqn is
linear, we see that any constant-
multiple of any soln is also a soln & also sum of two solns is also a soln. Thus, we see that the for

$$y(t) = C_1 e^{rt} + C_2 e^{r_2 t} \quad \text{--- (**)$$

is a soln to the diff’l eqn for any $C_1, C_2 \in \mathbb{R}$.

This soln (**) is called the general soln to the eqn & $C_1$ & $C_2$ can be calculated using the initial conditions $y(t_0) = y_0$ & $y'(t_0) = y_1$.

Ex: Find the soln to the diff’l eqn

$$\begin{bmatrix} y'' + 5y' + 6y = 0 \\ y(0) = 2 \quad y'(0) = 3 \end{bmatrix}$$
To solve this eqn we first need to find the general soln to the eqn

\[ y'' + 5y' + 6y = 0. \]

For that, we assume \( y = e^{rt} \) & get the characteristic eqn

\[ r^2 + 5r + 6 = 0. \]

Then we see

\[ (r^2 + 5r + 6) = (r+2)(r+3) = 0 \]

\[ \Rightarrow r = -2 \text{ or } r = -3. \]

Thus, the funs \( y_1 = e^{-2t} \) & \( y_2 = e^{-3t} \) are solns to the diff eqn & hence the general soln to the eqn is

\[ y(t) = C_1 e^{-2t} + C_2 e^{-3t}. \]
Now, checking the I.C, we get
\[ y(0) = 2 = C_1 e^0 + C_2 e^0 = C_1 + C_2 \]
\[ y'(0) = 3 = -2C_1 e^0 - 3C_2 e^0 = -2C_1 - 3C_2 \]

i.e. \[ 2 = C_1 + C_2 \quad \& \quad 3 = -2C_1 - 3C_2 . \]

Thus, solving for \( C_1 \) & \( C_2 \) we get
\[ C_1 = 9 \quad \& \quad C_2 = -7 . \]

\( \Rightarrow \) The soln to the initial value problem is
\[ y(t) = 9e^{-2t} - 7e^{-3t} . \]

Ex: Solve the initial value problem
\[ 4y'' - 8y' + 3y = 0 , \quad y(0) = 2 , \quad y'(0) = 1/2 . \]

Soln: Plugging in \( y = e^{rt} \), we get the characteristic eqn
\[ 4r^2 - 8r + 3 = 0. \]

\[ \Rightarrow (2r - 3)(2r - 1) = 0 \Rightarrow r = \frac{3}{2} \quad \text{or} \quad r = \frac{1}{2}. \]

Thus, \( y_1 = e^{\frac{3}{2}t} \) \& \( y_2 = e^{\frac{1}{2}t} \) are solns to the diff\'l eqn. Therefore, the general soln to the eqn is

\[ y(t) = C_1 e^{\frac{3}{2}t} + C_2 e^{\frac{1}{2}t} \]

& checking for initial conditions we get

\[ y(0) = 2 = C_1 + C_2 \]

\[ y'(0) = \frac{1}{2} = \frac{3}{2} C_1 + \frac{1}{2} C_2 \]

\[ \text{& solving for } C_1 \text{ & } C_2 \]

\[ C_1 = -\frac{1}{2} \quad \& \quad C_2 = \frac{5}{2}, \text{ that is, the} \]

\[ \text{soln to the initial value problem is} \]

\[ y(t) = -\frac{1}{2} e^{3t/2} + \frac{5}{2} e^{t/2}. \]
Chapter 3.2 & 3.3: Solns to linear eqns. Wronskian & Complex roots for characteristic eqn:

First, let's start with the general existence & uniqueness thm

**Theorem 3.2.1:**

Consider the initial value problem

\[ y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \]
\[ y'(t_0) = y'_0, \]

where \( p, q, g \) are cts on an open interval \( I \) that contains the point \( t_0 \). The problem has exactly one soln \( y = \phi(t) \), and the soln exists throughout the interval \( I \).
Moreover, if \( y_1 \) and \( y_2 \) are two solns to the eqn
\[
L[y] = y'' + p(t)y' + q(t)y = 0,
\]
and if \( y(t_0) = y_0 \) and \( y'(t_0) = y'_0 \) are given, then it is always possible to choose constants \( c_1 \) and \( c_2 \) so that
\[
y(t) = c_1 y_1(t) + c_2 y_2(t)
\]
satisfies the initial value problem (i.e., \( L[y] = 0 \) \& \( y(t_0) = y_0 \) \& \( y'(t_0) = y'_0 \)). If and only if the Wronskian
\[
W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 y'_2 - y'_1 y_2
\]
is not zero at \( t_0 \).
(The funs \( y_1 \) & \( y_2 \) are called the fundamental set of solns.)
Ex: Suppose \( y_1 = e^{rt} \) & \( y_2 = e^{st} \) are solns to \( L(y) = y'' + by' + cy = 0 \) & \( r_1 \neq r_2 \), then

\[
W[y_1, y_2] = \begin{vmatrix} e^{rt} & e^{st} \\ re^{rt} & re^{st} \end{vmatrix} = (r_2 - r_1)e^{(r_2-r_1)t}
\]

\( \to 0 \) for any \( t \in \mathbb{R} \).

There is also another theorem that calculates Wronskian without even knowing what the solns are. Namely

**Theorem 3.2.7: Abel's theorem:**

If \( y_1 \) & \( y_2 \) are solns to

\( L(y) = y'' + p(t)y' + q(t)y = 0 \) where \( p \) & \( q \) are cts on an open interval \( I \). Then

\( W[y_1, y_2](t) = c \cdot \exp (\int p(t) \, dt) \) where \( c \) depends only on \( y_1 \) & \( y_2 \), but not in \( t \).
Moreover, $W(y_1, y_2)(t)$ is either zero for all $t \in I$ (i.e., $c=0$) or never zero in $I$ ($c \neq 0$).

Now, let’s get back to the eqn

\[
\begin{bmatrix}
ay'' + by' + cy = 0 \\
y(t_0) = y_0 & y'(t_0) = y_0'
\end{bmatrix},
\]

and consider what happens when the characteristic eqn

\[ar^2 + br + c = 0\]

has two complex solns i.e., when $b^2 - 4ac < 0$.

Then we know that we have two roots to the characteristic eqn

\[r_1 = \lambda + i\mu \quad \text{and} \quad r_2 = \lambda - i\mu \quad \text{where} \quad \lambda, \mu \in \mathbb{R}.
\]
The nice thing is that if we consider the eqns in complex plane, then
\[ y_1 = e^{(\lambda + im)t} \quad \& \quad y_2 = e^{(\lambda - im)t} \]
are solns to the diff' eqn.

The not-so-nice thing is that these solns are complex valued fncts.

But we have some nice tools to find the "real" solns.

First thing we are going to use is Euler's formula that states
\[ e^{ix} = \cos x + i\sin x \quad \text{for} \quad x \in \mathbb{R}. \]

And the other information is the superposition of solns.

Namely:
We see \( y_1 = e^{(\lambda + i\mu)t} = e^{\lambda t + i\mu t} = e^{\lambda t} \cdot e^{i\mu t} \)
\[= e^{\lambda t} (\cos(\mu t) + i\sin(\mu t)); \quad \text{(Euler's formula)} \]
\[= e^{\lambda t} \cdot e^{-i\mu t} = e^{\lambda t} (\cos(\mu t) + i\sin(-\mu t)) \]
\[= e^{\lambda t} (\cos(\mu t) - i\sin(\mu t)); \quad \text{since } \sin(-x) = -\sin x. \]

But then, by linearity of the eqn, we see \( Y_1 = \frac{1}{2} (y_1 + y_2) = e^{\lambda t} \cos \mu t \) is a soln as well as \( Y_2 = \frac{1}{2i} (y_1 - y_2) = e^{\lambda t} \sin \mu t. \)

Therefore if we check \( W(Y_1, Y_2)(t), \)
we get
\[W(Y_1, Y_2) = \begin{vmatrix} e^{\lambda t} \cos \mu t & e^{\lambda t} \sin \mu t \\ e^{\lambda t} (\cos(\mu t) - \mu \sin(\mu t)) & e^{\lambda t} (\lambda \sin(\mu t) + \mu \cos(\mu t)) \end{vmatrix} = me^{2\lambda t}. \]
which is not zero for $\mu \neq 0$.

$\Rightarrow Y_1 = e^{\lambda t} \cos(\mu t)$ \& $Y_2 = e^{\lambda t} \sin(\mu t)$ form

the fundamental set of solns \& thus

the general soln is

$$y(t) = C_1 e^{\lambda t} \cos(\mu t) + C_2 e^{\lambda t} \sin(\mu t).$$

Ex. Solve the IVP

$$\begin{bmatrix}
y'' + 2y' + 2y = 0 \\
y(0) = 2, \ y'(0) = -2
\end{bmatrix}$$

Sln: We see that for this eqn, the characteristic eqn is

$$r^2 + 2r + 2 = 0.$$

Then we see $r^2 + 2r + 2 = (r^2 + 2r + 1) + 1 = 0$

$\Rightarrow (r + 1)^2 + 1 = 0 \Rightarrow r_1 = -1 + i \ & \ r_2 = -1 - i$ are

the solns to the characteristic eqn \&

(i.e. $\lambda = -1, \ \mu = 1$)
Thus the fundamental solns are
\[ y_1 = e^{-t} \cos t \quad \& \quad y_2 = e^{-t} \sin t. \]

\[ \Rightarrow \text{The general soln is} \]
\[ y(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t. \]

Checking the I.C. we get
\[ y(0) = c_1 e^{0} \cos 0 + c_2 e^{0} \sin 0 \]
\[ = c_1 = 2 \]
\[ \& \quad y'(t) = c_1 e^{-t}(-\cos t - \sin t) + c_2 e^{-t}(-\sin t + \cos t) \]
\[ \Rightarrow y'(0) = c_1 \cdot 1 \cdot (-1) + c_2 \cdot 1 \cdot (1) = c_2 - c_1 = -2 \]
\[ \Rightarrow c_2 = 0 \quad (\text{since} \ \ c_1 = 2) \]

\[ \Rightarrow \text{The soln to the IVP is} \]
\[ y(t) = 2e^{-t} \cos t. \]
Ex: Solve the IVP

\[
\begin{bmatrix}
y'' + 9y = 0 \\
y\left(\frac{\pi}{2}\right) = 1 \\
y'\left(\frac{\pi}{2}\right) = 1
\end{bmatrix}
\]

We see that the characteristic eqn for this eqn is

\[r^2 + 9 = 0 \Rightarrow \text{the roots are} \]

\[r_1 = 3i \quad \text{and} \quad r_2 = -3i.\]

\[\Rightarrow \lambda = 0 \quad \text{and} \quad \mu = 3. \text{ Therefore the fundamental solns are} \]

\[y_1(t) = \cos 3t \quad \text{and} \quad y_2(t) = \sin 3t\]

\[\Rightarrow \text{The general soln is} \]

\[y(t) = C_1 \cos 3t + C_2 \sin 3t.\]

Then checking the I.C. we get
\[ y\left(\frac{\pi}{2}\right) = C_1 \cos\left(\frac{3\pi}{2}\right) + C_2 \sin\left(\frac{3\pi}{2}\right) \]
\[ = -C_2 = 1 \quad \Rightarrow \quad C_2 = -1 \]

\[ y'(t) = -3C_1 \sin 3t + 3C_2 \cos 3t \]
\[ \Rightarrow y\left(\frac{\pi}{2}\right) = -3C_1 \sin\left(\frac{3\pi}{2}\right) + C_2 \cos\left(\frac{3\pi}{2}\right) \]
\[ = -3C_1 (-1) = 1 \]
\[ \Rightarrow C_1 = \frac{1}{3} \]

Thus, the solution to the IVP is

\[ y(t) = \frac{1}{3} \cos 3t - \sin (3t) \]

Chapter 3.4: Repeated roots & reduction of Order.

In the previous sections we saw how to solve 2nd order homogeneous constant coeff. differential eqns when when the characteristic eqn two distinct real or two distinct
complex roots. That method, however, gives us only one soln when we have repeated roots for the characteristic eqn. Nice thing is though, using that soln, we can find the second soln using what we call reduction of order. Which goes as follows:

Consider the eqn

\[ ay'' + by' + cy = 0 \] & that

\[ b^2 - 4ac = 0. \] Then the characteristic eqn

\[ ar^2 + br + c = 0 \] has only one repeated root

\[ r_1 = r_2 = \frac{-b}{2a}. \]

This implies that \[ y_1 = e^{\frac{-b}{2a}t} \] is a soln (up to this point, everything is exactly as before)

But now, we do not have the second soln,
and to find the second soln, \( y_2 \), we are going to assume that \( y_2 = v(t)y_1(t) \), that is, the second soln is first soln times another fun. (that is to say \( v(t) \) is going to be \( \frac{y_2}{y_1}(t) \)). Then we plug \( y_2 = v(t)y_1(t) \) into the eqn & solve for \( v(t) \).

Let's go back to our example.

\[ ay'' + by' + cy = 0 \quad \text{& let } y_1 \text{ is a soln. Then, we assume } y_2 = v(t)y_1 \quad \text{& get} \]

\[ y_2' = v'y_1 + vy_1' \quad \text{&} \]

\[ y_2'' = v''y_1 + v'y_1' + vv'y_1' + vy_1'' \]

\[ = v'y_1 + 2vv'y_1' + vy_1'' \]

Thus, plugging these into the eqn we should get 0; i.e.
\[ ay''_2 + by'_2 + cy_2 = 0 \]

\[ a(\sqrt{y_1} + 2\sqrt{y'}_1 + \sqrt{y''}_1) + b(\sqrt{y'_1} + \sqrt{y''}_1) + cy_1 = 0 \]

Collect \( \sqrt{y'_1} \) terms

\[ a\sqrt{y_1} + 2a\sqrt{y'}_1 + b\sqrt{y''}_1 + \sqrt{(ay''_1 + by'_1 + cy_1)} = 0 \]

\[ = 0 \text{ since } y_1 \text{ is a soln to the eqn.} \]

\[ \Rightarrow a\sqrt{y'_1} + 2a\sqrt{y'}_1 + b\sqrt{y''}_1 = 0. \]

(This method is called reduction of order since by calling \( \sqrt{\cdot} = f \), we can reduce this second order eqn in \( \sqrt{\cdot} \) into a first order eqn in \( f \), and of course once we find \( f \), we integrate it to get \( \sqrt{\cdot}(t) \)
Now recall that $y_1 = e^{-\frac{b}{2a}t}$ & if we plug that in, we get

$$a\sqrt{e}^{-\frac{b}{2a}t} + 2a\sqrt{e}^{-\frac{b}{2a}t} + b\sqrt{e}^{-\frac{b}{2a}t} = 0$$

$$= 0$$

$$\Rightarrow a\sqrt{e}^{-\frac{b}{2a}t} \to 0 \Rightarrow \sqrt{e}^{-\frac{b}{2a}t} \to 0 \Rightarrow \text{integrating twice we get}$$

$$\sqrt{e} = c_1 + c_2t , c_1, c_2 \in \mathbb{R}.$$  

i.e. $y_2(t) = (c_1 + c_2t)e^{-\frac{b}{2a}t}$

$$= c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t}$$

this is already a soln.

i.e. $\sqrt{e} = c_1 + c_2t$ is already a soln. Now, let's check the Wronskian to see whether $y_1$ & $y_2$ form a
Fundamental set of solns.

$$W[y_1, y_2] = \begin{vmatrix} e^{-\frac{b}{2a}t} & e^{-\frac{b}{2a}t} \\ -\frac{b}{2a}e^{\frac{b}{2a}t} & (1-\frac{bt}{2a})e^{-\frac{bt}{2a}} \end{vmatrix} = e^{-\frac{bt}{2a}}$$

which is never zero.

Thus, the general soln is

$$y(t) = c_1 e^{\frac{bt}{2a}} + c_2 t e^{\frac{bt}{2a}}.$$ 

**Remark:** This method of reduction of order can also be used in eqns of the form

$$y'' + p(t)y' + q(t)y = 0$$

and that we know a soln $y_1$, not everywhere zero. But in this case, the eqn reduces to

$$y_1 y'' + (\dot{y}_1 + p(t)y_1) \dot{y}' = 0.$$
Then we let $r = \frac{1}{2}$ and solve this eqn.

Ex: Find the soln of the IVP

\[
\begin{bmatrix}
y'' - y' + \frac{y}{4} = 0 \\
y(0) = 2, \ y'(0) = \frac{1}{3}
\end{bmatrix}
\]

We see that the characteristic eqn is

\[ r^2 - r + \frac{1}{4} = 0 \]

\[ \implies (r - \frac{1}{2})^2 = 0 \implies \text{we have the repeated root } r = \frac{1}{2}. \]

\[ \implies y_1 = e^{\frac{1}{2}t} \quad \text{and} \quad y_2 = te^{\frac{1}{2}t} \] are solns to the homogeneous eqn & the general soln is

\[ y(t) = c_1 e^{\frac{1}{2}t} + c_2 te^{\frac{1}{2}t} \]

Now, checking the initial data, we get

\[ y(0) = c_1 = 2 \quad \text{and} \quad y'(t) = \frac{c_1}{2} e^{\frac{1}{2}t} + c_2 e^{\frac{1}{2}t} \left(1 + \frac{t}{2}\right) \]
\[ y'(0) = \frac{c_1}{2} + c_2 = \frac{1}{3} \implies c_2 = -\frac{2}{3}. \]

\[ \implies y = 2e^{\frac{t^2}{2}} - \frac{2}{3}te^{\frac{t^2}{2}} \text{ is the soln to the IVP.} \]

\textbf{Ex.} Given that \( y_1(t) = t^{-1} \) is a soln of \( 2t^2y'' + 3ty' - y = 0, \ t > 0, \) find the fundamental set of solns.

In this question, since we are given one soln, we can use reduction of order & set \( \tilde{y}_2 = v(t) \cdot t^{-1} \) & get
\[ \tilde{y}_2' = v' \cdot t^{-1} - v \cdot t^{-2} \& \]
\[ \tilde{y}_2'' = v'' \cdot t^{-1} - 2v' \cdot t^{-2} + 2vt^{-3} \& \text{plugging these into the eqn we get} \]
\[ 2t^2 \left( v'^2 - 2v't^2 + 2vt^2 \right) + 3t \left( v't^{-1} - vt^2 \right) - vt^{-1} = 0 \]

\[ \Rightarrow 2t v'' + \left( -4 + 3 \right) v' + \left( 4t^{-1} - 3t^2 - t^3 \right) v = 0 \]

\[ \Rightarrow 2t v'' - v' = 0. \]

Now, let \( v' = f \) & get

\[ 2t f' - f = 0 \quad \text{(or) } f' - \frac{1}{2t} f = 0. \]

This is a first order separable eqn i.e we can either use integrating factor or separability of this eqn & get

\[ \frac{f'}{f} = \frac{1}{2t} \quad \Rightarrow \quad \frac{df}{f} = \frac{dt}{2t} \quad \& \]

integrating both sides we get

\[ \ln |f| = \frac{1}{2} \ln |t| + C \quad \text{CE12} \]

\[ \Rightarrow \quad f = C \cdot e^{\frac{1}{2} \ln |t|} = C \cdot e^{\ln t^{\frac{1}{2}}} \quad \text{CE2} \]

\[ = C \cdot t^{\frac{1}{2}}. \quad \text{(recall that } t > 0) \]

\[ \Rightarrow \quad v = \int f \, dt = \frac{2}{3} C e^{3t^{\frac{1}{2}}} + k, \quad k \in \mathbb{R}. \]
\[ y_2 = \sqrt{t} t^{-1} = \frac{2}{3} G t^{-1/2} + k t^{-1} \] is also a soln since the second term in the sum is just \( k \cdot y_1 \), we can drop this term and divide the rest by \( \frac{2}{3} G \) & still get a soln i.e \( y_2 = t^{1/2} \) is a soln. Moreover, if we check the Wronskian, we get

\[
W(y_1, y_2)(t) = \begin{vmatrix} t^{-1} & t^{1/2} \\ -t^{-2} & \frac{1}{2} t^{-1/2} \end{vmatrix} = \frac{3}{2} t^{-3/2} \neq 0 \text{ for } t > 0
\]

i.e \( y_1 = t^{-1} \) & \( y_2 = t^{1/2} \) form the fundamental set of solns. Therefore the general soln to the eqn is

\[ y(t) = C_1 t^{-1} + C_2 t^{1/2} \]
Chapter 3.5: Nonhomogeneous Eqns & Method of Undetermined Coefficients

In this chapter we are going to look at the nonhomogeneous eqns

\[ L(y) = y'' + p(t)y' + q(t)y = g(t) \]

where \( p, q \) & \( g \) are given at its fns on the open interval \( I \).

The eqn \( y'' + p(t)y' + q(t)y = 0 \) is called the homogeneous diff eqn corresponding to the eqn \( L(y) \).

Then we can easily verify that if \( Y_1 \) & \( Y_2 \) are two solns to the nonhomogeneous eqn, then \( Y_1 - Y_2 \) is a soln to the homogeneous part of
the eqn since the eqn is linear.

Namely if

\[ y_1'' + p(t)y_1' + q(t)y_1 = g(t) \quad \& \quad y_2'' + p(t)y_2' + q(t)y_2 = g(t) \]

as well, then

\[ (y_1 - y_2)'' + p(t)(y_1 - y_2)' + q(t)(y_1 - y_2) = 0. \]

i.e. \( y_1 - y_2 \) solve the homogeneous part of the eqn. Then we see that the general soln to the nonhomogeneous eqn \( L(y) = y'' + p(t)y' + q(t)y = g(t) \) can be written as

\[ y = c_1 y_1(t) + c_2 y_2(t) + \gamma(t) \]

where \( c_1, c_2 \in \mathbb{R} \) and \( y_1, y_2 \) form the fundamental set of solns to the homogeneous part of the eqn, i.e.

\[ y'' + p(t)y' + q(t)y = 0. \]
So, to solve nonhomogeneous eqns, we first find the general soln, \( C_1 y_1 + C_2 y_2 \), to the homogeneous eqn corresponding to the nonhomogeneous eqn. This soln is frequently called the complementary soln.

Then we find ANY soln \( Y(t) \) that satisfies the nonhomogeneous eqn. This \( Y \) is often called a particular soln.

Then we take the sum of the complementary & particular solns.

In this chapter we are going to learn the method of undetermined coefficients where we guess the form of the particular soln but leave the coefficients unspecified. If the form we assumed doesn't work we
can try to modify our guess & try again.
The nice thing about this method is that it
is straightforward, but is usually pretty
limited to those eqns where we can easily
guess what the particular soln should
look like.

Let's see how we can use this method
in examples.

Ex: Find a particular soln of

\[ y'' - 3y' - 4y = 3e^{2t} \]

In this example, since \( y(t) = 3e^{2t} \) &
since the derivatives & integrals of \( Ce^{2t} \)
is also of the form \( De^{2t} \), we guess
that the soln is of the form (let me
mention again, that the soln does not have to be in this form, this is a guess)

\[ Y(t) = A e^{2t}. \]  Now, plugging this into the eqn we get

\[ Y''(t) - 3Y'(t) - 4Y(t) = 3e^{2t} \]

\[ \Rightarrow 4A e^{2t} - 6A e^{2t} - 4A e^{2t} = 3e^{2t} \]

\[ \Rightarrow -6A e^{2t} = 3e^{2t} \Rightarrow A = -\frac{1}{2}. \]

\[ \Rightarrow Y(t) = -\frac{1}{2} e^{2t} \text{ is a particular soln.} \]

\[ \boxed{\text{Ex: Find a particular soln of}} \]

\[ y'' - 3y' - 4y = 2\sin t. \]

In this example, we can try to do the same thing and assume \[ Y(t) = A \sin t, \]
we see that \[ Y'' \] \& \[ Y \] gives us a multiple
of $\sin t$ whereas $Y'$ gives us cost. This implies that assuming $Y(t) = A\sin t$
is not going to be enough & we need to take into account of this byproduct "cost". We can do this by assuming a
soln of the form

$$Y(t) = A\sin t + B\cos t.$$ Then plugging this into the eqn we get

$$Y'' - 3Y' - 4Y = 2\sin t$$

$$\Rightarrow (A\sin t + B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t \quad \text{(collecting } \sin t \text{ & } \cos t \text{ terms)}$$

$$\Rightarrow \sin t (-A + 3B - 4A) + \cos t (-3B - 3A - 4B) = 2\sin t.$$ 

$$\Rightarrow \sin t (3B - 5A) + \cos t (-5B - 3A) = 2\sin t.$$ [0]

$$\Rightarrow 3B - 5A = 2 \quad \text{and} \quad -5B - 3A = 0$$

$$\Rightarrow A = \frac{-5}{17}, \quad B = \frac{3}{17}.$$
A particular soln is
\[ Y(t) = \frac{-5}{17} \sin t + \frac{3}{17} \cos t. \]

Ex: Find a particular soln of
\[ y'' - 3y' - 4y = -8e^t \cos(2t). \]

In this problem, similar to the arguments above, we are going to assume
\[ Y(t) = A e^t \cos(2t) + B e^t \sin(2t) \]
& if we plug this into the eqn & solve for A & B we get (I leave this calculation as an exercise) \( A = \frac{10}{13}, \ B = \frac{2}{13}. \)

Now suppose that \( g(t) \) in the nonhomogeneous eqn is of the form \( g(t) = g_1(t) + g_2(t) \) and suppose that \( Y_1 \) & \( Y_2 \) are solns to the
eqns \[ ay'' + by' + cy = g_1(t) \quad \& \quad ay'' + by' + cy = g_2(t) \]
respectively, then \( Y_1 + Y_2 \) is a soln to the eqn \[ ay'' + by' + cy = g(t) \].

Ex: Find a particular soln of \[ y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos(2t) \].

In this question, we see that \( g(t) = 3e^{2t} + 2\sin t - 8e^t \cos(2t) \)

\[ = g_1(t) + g_2(t) + g_3(t) \].

Thus a particular soln to this eqn is a sum of particular solns we got from the eqns
\[ y'' - 3y' - 4y = g_i(t) \quad i \in \{1,2,3\} \]
That is, a particular soln is
\[ y(t) = \frac{1}{2} e^{2t} + \frac{3}{17} \cos t - \frac{5}{17} \sin t + \frac{10}{13} e^{t} \cos (2t) + \frac{2}{13} e^{t} \sin (2t). \]

Now, if we take a look at the homogeneous part of this eqn,
\[ y'' - 3y' - 4y = 0, \]
we get the characteristic eqn
\[ r^2 - 3r - 4 = (r-4)(r+1) = 0 \]
\[ \Rightarrow r_1 = -1 \quad \text{and} \quad r_2 = 4. \] Thus the fundamental set of solns to the homogeneous eqn are \( y_1 = e^{4t} \) and \( y_2 = e^{-t} \).

But then we see that if we are given \( g(t) = M \cdot e^t \), \( M \in \mathbb{R} \),
we cannot just assume that the
particular soln has the form
\[ Y(t) = Ae^{-t} \] since any multiple of \( e^{-t} = y_2 \) solves the homogeneous eqn; i.e., such \( Y(t) \) gives
\[ Y'' - 3Y' - 4Y = 0 \ (\neq y(t)). \]

But even in these cases we can modify our guess & use the method of undetermined coefficients.

To see how we can modify our soln, let's look at an easier example: Consider the eqn
\[ y' + y = 2e^{-t}. \]

Then, we see the homogeneous eqn \( y' + y = 0 \) has the soln \( y_h(t) = e^t \) simply by separating variables & getting
\[
\frac{dy}{y} = -dt \Rightarrow \ln|y| = -t + C \quad \text{CEIR}
\]
\[
\Rightarrow y = Ce^{-t} \quad \text{CEIR}
\]

Then we see that our choice of \( y(t) = Ae^{-t} \) for the homogeneous eqn is not going to work. But, if we solve this eqn using integrating factor, we get
\[
\mu(t) = e \int 1 dt = e^t
\]
\[
\Rightarrow e^t y' + e^t y = 2e^t e^{-t} = 2
\]
\[
\Rightarrow (e^t y)' = 2 \Rightarrow \text{integrating both sides, we get}
\]
\[
e^t y = 2t + C \Rightarrow y = 2te^{-t} + Ce^{-t}
\]

This easy example suggests that
If we have the eqn
\[ y'' - 3y' - 4y = 3e^{-t}, \]
since the homogeneous solns are
\[ y_1 = e^{4t} \quad \text{and} \quad y_2 = e^{-t}, \]
since \( g(t) = 3e^{-t} \), we guess the particular soln to have the form
\[ Y(t) = A \cdot t \cdot e^{-t}. \]
Then, if we plug this into the eqn, we get
\[ Y'(t) = Ae^{-t} - Ate^{-t} \quad \text{and} \]
\[ Y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{t} = Ae^{-t} - 2Ae^{-t}. \]
\[ \Rightarrow 1 - 3Y' - 4Y = 3e^{-t} \]
\[ = (Ate^{-t} - 2Ae^{-t}) - 3(Ae^{-t} - Ate^{-t}) \]
\[ -4(Ate^{-t}) = 3e^{-t} \]
\[ \Rightarrow e^{-t}(-2A - 3A) + te^{-t}(A + 3A - 4A)te^{-t} = 3e^{-t} \]
\[ \Rightarrow -5A = 3 \Rightarrow A = \frac{-3}{5} . \]

Therefore, \( Y(t) = \frac{-3}{5} te^{-t} \) is a particular soln.

Some Particular Solns of \( ay'' + by' + cy = g(t) \):

- \( g(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n \)
  \[ \Rightarrow Y(t) = \sum^s (A_0 t^n + A_1 t^{n-1} + \cdots + A_n) \]
- \( g(t) = e^{\alpha t} (a_0 t^n + a_1 t^{n-1} + \cdots + a_n) \)
  \[ \Rightarrow Y(t) = \sum^s (A_0 t^n + A_1 t^{n-1} + \cdots + A_n) e^{\alpha t} \]
- \( g(t) = (a_0 t^n + a_1 t^{n-1} + \cdots + a_n) e^{\alpha t} \sin(\beta t) \) or
  \( (a_0 t^n + a_1 t^{n-1} + \cdots + a_n) e^{\alpha t} \cos(\beta t) \)
  \[ \Rightarrow Y(t) = \sum^s \left( (A_0 t^n + A_1 t^{n-1} + \cdots + A_n) e^{\alpha t} \cos(\beta t) \right. \\
  \left. + (B_0 t^n + B_1 t^{n-1} + \cdots + B_n) e^{\alpha t} \sin(\beta t) \right) \]

where \( s = 0, 1 \) or \( 2 \).
Ex: Solve the IVP

\[
\begin{bmatrix}
y'' - 2y' + y = 2e^t \\
y(0) = 1, \ y'(0) = 1
\end{bmatrix}
\]

To solve this eqn, we need to:

1) Find the fundamental set of solns to the homogeneous eqn \( y, x y, \) & write the complementary soln \( y = c_1 y + c_2 y_z, c_1, c_2 \in \mathbb{R}. \)

2) Find a particular soln \( Y(t) \) that satisfies \( y'' - 2y' + y = 2e^t. \)

3) Write the general soln \( y(t) = c_1 y + c_2 y_z + Y(t) \) & check I.C. to solve for \( c_1, c_2. \)
Now, let's start with step 1) & find the solns to

\[ y'' - 2y' + y = 0. \]

This is a 2nd order linear homogeneous constant coeff. diff'l eqn. Then we plug in \( y = e^{rt} \) & get the characteristic eqn

\[ r^2 - 2r + 1 = (r-1)^2 = 0. \]

\( \Rightarrow r = 1 \) is a double root & thus we have the solns

\[ y_1(t) = e^{rt} = e^t \quad \& \quad y_2(t) = t \cdot e^t. \]

\( \Rightarrow F(t) = c_1 e^t + c_2 t e^t \) is the general soln to the homogeneous eqn.
Now, we need to find a particular soln. We see that $g(t) = 2e^t$ and that $e^t$ is a soln to the homogeneous eqn. This means that we would look for gens of the form $At e^t$ but $te^t$ is also a soln. This implies that we should look for gens of the form $\gamma(t) = At^2 e^t$. Then we see

\[
\gamma'(t) = 2Ate^t + At^2 e^t
\]

\[
\gamma''(t) = 2Ae^t + 2Ate^t + 2Ate^t + At^2 e^t
\]

\[
= 2Ae^t + 4Ate^t + At^2 e^t.
\]

\[
\gamma'' - 2\gamma' + \gamma = 2Ae^t + 4Ate^t + At^2 e^t
\]

\[
-2(2Ae^t + At^2 e^t) + At^2 e^t
\]

\[
= 2e^t
\]
\[ 2Ae^t = 2e^t \Rightarrow A = 1. \]
\[ \Rightarrow Y(t) = te^t \text{ is a particular soln.} \]

\[ \Rightarrow \text{The general soln is} \]
\[ Y(t) = c_1y_1 + c_2y_2 + Y(t) \]
\[ = c_1e^t + c_2te^t + te^t \]

\[ \Rightarrow \text{checking the I.C. we get} \]
\[ y(0) = c_1 = 1 \quad \& \]
\[ y'(0) = e^t + c_2e^t + c_2te^t + 2te^t + te^t \]
\[ \Rightarrow y'(0) = 1 + c_2 = 1 \Rightarrow c_2 = 0 \]

\[ \Rightarrow \text{The soln to the IVP is} \]
\[ Y(t) = e^t + te^t. \]

\[ \underline{Ex:} \text{ Solve the IVP} \]
\[
\begin{bmatrix}
y'' + 6y' + 10y = t^2 + 2 \\
y(0) = 0, y'(0) = 0
\end{bmatrix}
\]
For this eqn, again, we first find the solns to the homogeneous eqn
\[ y'' + 6y' + 10y = 0. \]
we see that the characteristic eqn of this homogeneous eqn is
\[ r^2 + 6r + 10 = 0. \]
Then we see that the roots are
\[ r_{1,2} = \frac{-6 \pm \sqrt{36 - 40}}{2} = -3 \mp \sqrt{4} \]
\[ = -3 \mp 2i. \]

⇒ The fundamental set of solns are
\[ y_1(t) = e^{-3t} \cos(2t) \quad \& \quad y_2(t) = e^{-3t} \sin(2t) \]

Now, we want to find a particular soln to the eqn. Since \[ g(t) = t^2 + 2 \]
we expect particular soln to have the form

\[ Y(t) = At^2 + Bt + C. \]

Then we see \( Y' = 2At + B \) \& \( Y'' = 2A. \)

\[ Y'' + 6Y' + 10Y = \underbrace{2A} + \underbrace{12At} + \underbrace{6B} + \underbrace{10At^2} + \underbrace{10B} + \underbrace{10C} = t^2 + 2 \]

\[ \Rightarrow \text{ Collecting powers of } t \text{ together, we get} \]

\[ 10At^2 + (12A + 10B)t + 2A + 6B + 10C = t^2 + 2 \]

\[ \Rightarrow 10A = 1 \Rightarrow A = \frac{1}{10} \]

\[ 12A + 10B = 0 \Rightarrow B = \frac{-12A}{10} = \frac{-12}{100} \]

\[ 2A + 6B + 10C = 2 \Rightarrow \frac{1}{5} - \frac{72}{100} + 10C = 2 \]

\[ \Rightarrow 10C = \frac{252}{100} \Rightarrow C = \frac{252}{1000} \]
Thus, $y(t) = \frac{1}{10} t^2 - \frac{12}{100} t + \frac{252}{100}$ is a particular soln.

Therefore, the general soln is

$y(t) = c_1 y_1 + c_2 y_2 + y$

$= c_1 e^{-3t} \cos(2t) + c_2 e^{-3t} \sin(2t)$

$+ \frac{1}{10} t^2 - \frac{12}{100} t + \frac{252}{100}$

Now, checking the initial conditions

$y(0) = 1 \Rightarrow c_1 + \frac{252}{100} = 0 \Rightarrow c_1 = -\frac{252}{100}$

$y'(t) = -3c_1 e^{-3t} \cos(2t) - 2c_1 e^{-3t} \sin(2t)$

$- 3c_1 e^{-3t} \sin(2t) + 2c_2 e^{-3t} \cos(2t)$

$+ \frac{1}{5} t - \frac{12}{100}$

$= -3c_1 + 2c_2 - \frac{12}{100} = 0$
\[ 2c_2 = \frac{12}{100} + 3c_1 = \frac{12}{100} - \frac{756}{100} \]

\[ c_2 = \frac{-372}{100} = -3.72 \]

\[ y(t) = -\frac{252}{100} e^{-3t} \cos(2t) - \frac{372}{100} e^{-3t} \sin(2t) \]

\[ + \frac{1}{60} t^2 - \frac{12}{100} t + \frac{252}{100} \]

is the soln to the IVP.

Ex: Find a particular soln of

\[ y'' - y' = t^3 \]

In this example we see that \( g(t) = t^3 \)

& thus, we would like to look for solns of the form

\[ \tilde{y}(t) = At^3 + Bt^2 + Ct + D \]
But, since the eqn doesn't have an explicit \( y \) dependence ( \( y \) terms start from the first derivative), we see that if we plug in

\[ Y(t) = At^3 + Bt^2 + Ct + D, \]

the highest order \( t \) term we are going to get is \( t^2 \) which is a problem.

Thus, we need to look for a particular soln of the form

\[ Y(t) = At^4 + Bt^3 + Ct^2 + Dt + E \]

or (since the constant \( E \) doesn't change the value of the eqn)

\[ Y(t) = At^4 + Bt^3 + Ct^2 + Dt \]

\[ = t(At^3 + Bt^2 + Ct + D). \]
Now, plugging in this $Y(t)$ into the eqn, we get

$Y'(t) = 4At^3 + 3Bt^2 + 2ct + D$

$Y''(t) = 12At^2 + 6Bt + 2C$

$\Rightarrow Y'' - Y' = -4At^3 + (12A - 3B)t^2 + (6B - 2C)t + 2C - D$

$= t^3$

$\Rightarrow -4A = 1, \quad 12A - 3B = 0$

$6B - 2C = 0 \quad \& \quad 2C - D = 0$

$\Rightarrow A = \frac{-1}{4}, \quad B = 4A = -1$

$c = 3B = -3 \quad \& \quad D = 2C = -6$

$\Rightarrow Y(t) = -\frac{1}{4} t^4 - t^3 - 3t^2 - 6t$

Chapter 3.6: Variation of Parameters:

In the previous chapter we saw
how to find a particular soln to a nonhomogeneous eqn when we have nonhomogeneity facts that makes it easier for us to guess what a particular soln should look like.

In this chapter we are going to see what we can do when we don't have such facts $g(t)$ on the RHS. The method we are going to use is called variation of parameters & it goes as follows:

Consider the general 2nd order nonhomogeneous eqn

$$y'' + p(t)y' + q(t)y = g(t), \quad \text{--- (1)}$$

and assume that we know the complementary soln (soln to the homogeneous
part of the eqn),

\[ y_c = c_1 y_1(t) + c_2 y_2(t) \quad c_1, c_2 \in \mathbb{R} \]

Then, we assume a soln to the diff' eqn (x) to have the form

\[ y(t) = u_1(t) y_1(t) + u_2(t) y_2(t) \]

where \( u_1 \) & \( u_2 \) are fncs (This looks like the complementary soln where the constants \( c_1 \) & \( c_2 \) are replaced by fncs \( u_1(t) \) & \( u_2(t) \)).

Then if we plug this into the eqn, we get

\[ y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2' \]

at this step we further require

\[ u_1' y_1 + u_2' y_2 = 0 \]

(this requirement is going to be clear in the calculations)
Thus, we see
\[ y' = u_1 y_1' + u_2 y_2' \quad \text{and thus} \]
\[ y'' = u_1 y_1'' + u_1 y_1' + u_2 y_2' + u_2 y_2'' \]
Then, plugging these into the eqn we get
\[ y'' + p(t)y' + q(t) = g(t) \]
\[ \implies u_1 y_1'' + u_1 y_1' + u_2 y_2' + u_2 y_2'' + p(t)(u_1 y_1' + u_2 y_2') + q(t)(u_1 y_1 + u_2 y_2) = g(t) \]

Now, collecting \( u_1 \) & \( u_2 \) terms together, we get
\[ u_1(y_1'' + p(t)y_1' + q(t)y_1) + u_2(y_2'' + p(t)y_2' + q(t)y_2) + u_1 y_1' + u_2 y_2' = g(t) \]

The first & the second sums are zero since \( y_1 \) & \( y_2 \) satisfy the homogeneous
diff'1 eqn. Thus

\[ u'_1 y'_1 + u'_2 y'_2 = g(t) \]

putting this together with our previous assumption, we get

\[
\begin{bmatrix}
  u'_1 y'_1 + u'_2 y'_2 = 0 \\
  u'_1 y'_1 + u'_2 y'_2 = g(t)
\end{bmatrix}
\]

But now, this can be seen as

\[
\begin{bmatrix}
  y_1(t) & y_2(t) \\
  y'_1(t) & y'_2(t)
\end{bmatrix}
\begin{bmatrix}
  u'_1 \\
  u'_2
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  g(t)
\end{bmatrix}
\]

and as long as \[ \begin{bmatrix}
  y_1 & y_2 \\
  y'_1 & y'_2
\end{bmatrix}(t) \] is invertible,

\[ \begin{vmatrix}
  y_1 & y_2 \\
  y'_1 & y'_2
\end{vmatrix}(t) = W(y_1, y_2)(t) \neq 0, \]

we can uniquely solve for the system
for \( u_1' \) & \( u_2' \) & get

\[
\begin{align*}
u_1' &= \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)} \quad & u_2' &= \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}
\end{align*}
\]

\[ \Rightarrow u_1 = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} \, dt + c_1 \quad & u_2 = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \, dt + c_2 \]

Then, plugging these into the eqn

\[ y(t) = u_1 \, y_1 + u_2 \, y_2 \]

we find the general soln to the eqn

\[ y'' + p(t)\, y' + q(t)\, y = g(t) \]

I can hear the questions that you are thinking, how can we make sure
that the requirement
\[ u_1'y_1 + u_2'y_2 = 0 \]
is enough to make things work? That is a valid and good question. The following theorem guarantees us that this method actually works.

**Theorem 3.6.1:** Consider the nonhomogeneous second order linear differential equation
\[ y'' + p(t)y' + q(t)y = g(t). \]

If \( p, q, g \) are continuous on an open interval \( I \) and if the functions \( y_1, y_2 \) form a fundamental set of solutions to the homogeneous part of the equation
\[ y'' + p(t)y' + q(t)y = 0, \] then
a particular soln to the eqn is

\[ Y(t) = -y_1 \int_0^t \frac{y_2(s) g(t)}{W(y_1, y_2)(s)} \, ds + y_2 \int_0^t \frac{y_1(s) g(s)}{W(y_1, y_2)(s)} \, ds \]

where to is any chosen pt. in \( I \). Then the general soln is

\[ Y = C_1 y_1 + C_2 y_2 + Y(t). \]

Ex: Find the general soln of

\[ y'' + 4y = 8 \tan t \quad -\frac{\pi}{2} < t < \frac{\pi}{2}. \]

We see that the RHS, \( g(t) \), is equal to \( 8 \tan t \) so it doesn't fit our method of undetermined coefficients scheme. Here we need to use the variation of parameters. But first, we need to find the
Solutions to the homogeneous eqn
\[ y'' + 4y = 0. \]

We see that this is a 2nd order constant coefficient linear diff' eqn. Then we plug in \( y = e^{rt} \) & get the char. eqn
\[ r^2 + 4 = 0 \]
\[ \Rightarrow r = \pm 2i. \]

Thus, the fundamental set of solutions to this homogeneous eqn are
\[ y_1 = \cos(2t) \quad \text{&} \quad y_2 = \sin(2t). \]

Then, we assume the soln to the nonhomogeneous eqn has the form
\[ y(t) = u_1 y_1 + u_2 y_2 \quad \text{& also} \]
we assume that

\[ u_1', y_1 + u_2', y_2 = 0 \quad \& \quad \text{get} \]

\[ u_1' = \frac{-y_2 g}{W(y_1, y_2)} \quad \& \quad u_2' = \frac{y_1 g}{W(y_1, y_2)} \]

Calculating \( W(y_1, y_2) \) we get

\[
W(y_1, y_2)(t) = \begin{vmatrix}
\cos(2t) & \sin(2t) \\
-2\sin(2t) & 2\cos(2t)
\end{vmatrix} = 2
\]

\[ \Rightarrow u_1' = -\frac{\sin(2t) \cdot 8 \tan t}{\sqrt{2}} \Rightarrow u_1' = -4 \tan t \sin 2t \]

\[ \Rightarrow u_1' = -8 \frac{\sin t}{\cos t} \sin t \cos t = -8 \sin^2 t \]

& \[ u_2'(t) = \frac{8 \tan t \cdot \cos(2t)}{2} = 4 \frac{\sin t}{\cos t} (2 \cos^2 t - 1) \]

\[ = 4 \sin t \left( 2 \cos t - \frac{1}{\cos t} \right) \]
\[ u_1 = \int -8 \sin^2 t \, dt \]

Half angle formula

\[ = 8 \int \frac{1}{2} (\cos(2t) - 1) \, dt \]

\[ = 4 \left( \int \cos 2t \, dt - \int dt \right) = 2 \sin 2t - 4t + C_1 \]

\[ = 4 \sin t \cos t - 4t + C_1, \quad C_1 \in \mathbb{R} \]

& \quad u_2 = 4 \int \sin t \left( 2 \cos t - \frac{1}{\cos^2 t} \right) \, dt

\[ = 8 \int \sin t \cos t \, dt - 4 \int \frac{\sin t}{\cos t} \, dt \]

\[ = -4 \cos^2 t + 4 \ln |\cos t| + C_2 \quad (\text{on } \left( \frac{-\pi}{2}, \frac{\pi}{2} \right) \cap \cos t > 0) \]

\( \Rightarrow \) The general solution is

\[ y(t) = (4 \sin t \cos t - 4t) \cos 2t + (4 \ln \cos t - 4 \cos^2 t) \sin 2t + C_1 \cos 2t + C_2 \sin 2t. \]
This also implies that a particular soln is

$$y(t) = (4\sin t \cos t - 4t) \cos 2t$$

$$+ (4\ln(\cos t) - 4 \cos^2 t) \sin(2t).$$

Ex: Find the general soln to the eqn

$$y'' + 4y' + 4y = t^{-2} e^{-2t} \quad t > 0.$$

We are going to use variation of parameters in this question. For that, we need to find the solns to the homogeneous eqn

$$y'' + 4y' + 4y = 0$$

and we see that char. eqn for this eqn is

$$r^2 + 4r + 4 = 0.$$

$$\Rightarrow r = -2 \text{ is a double root}.$$
\[ \Rightarrow \text{The solutions to the homogeneous eqn are } y_1 = e^{-2t} \text{ & } y_2 = te^{-2t}. \text{ Thus,} \]

\[
W(y_1, y_2)(t) = \begin{vmatrix}
e^{-2t} & te^{-2t} \\
-2e^{-2t} & e^{-2t} - 2te^{-2t}
\end{vmatrix} = e^{-4t}
\]

\[ \Rightarrow u_1' = -\frac{y_2 g}{W(y_1, y_2)} = -\frac{te^{-2t} \cdot t e^{-2t}}{e^{-4t}} = -t^{-1} \Rightarrow u_1 = -\int \frac{1}{t} \, dt = -\ln t + C_1 \quad (t>0) \]

\[ u_2' = \frac{y_1 g}{W(y_1, y_2)} = \frac{e^{-2t} \cdot t e^{-2t}}{e^{-4t}} = t^{-2} \]

\[ \Rightarrow u_2 = \int t^{-2} \, dt = -\frac{1}{t} + C_2 \quad C_1, C_2 \in \mathbb{R} \]

\[ \Rightarrow \text{The general soln is} \]
\[ y(t) = -\ln t e^{-2t} - \frac{1}{t} te^{-2t} + c_1 e^{-2t} + c_2 te^{-2t} \\
= -\ln t e^{-2t} - e^{-2t} + c_1 e^{-2t} + c_2 te^{-2t} \\
= -\ln t e^{-2t} + c_3 e^{-2t} + c_2 te^{-2t} \\
\]

\[ c_2, c_3 \in \mathbb{R} \]

\[ \Rightarrow Y(t) = -\ln t e^{-2t} \text{ is a particular soln.} \]

**Chapter 3.7: Mechanical & Electrical Vibrations**

In this chapter we are going to look at eqns that govern the basic behaviour of a spring & electrical circuits.

In a spring system we have

\[ \text{spring at rest} \quad \frac{e}{\text{spring}} \quad \frac{\text{mass } m}{\text{spring with a mass } m \text{ attached, at rest}} \]

\[ (1) \quad \text{---} \quad \text{---} \quad (2) \]
& we can move the spring away from its resting position by a amount & get

\[ F_s = -k|u(t)u| \]

Then, we have several forces acting on the mass \( m \):

1) \( F_s = -k|u(t)u| \) which is Hooke's law

where \( k \) is the proportionality constant of the spring, \( k \in \mathbb{R}, k > 0 \)

2) Gravitational force \( mg \), where at rest gives us \( F_c = mg \) & we see from (2), \( mg - kL = 0 \).

3) Some viscous damping force, which is a resistive force that is proportional to the velocity but in the opposite direction i.e.

\[ F_d = -\delta u'(t) \] where \( \delta \in \mathbb{R}, \delta > 0 \).
Then, writing Newton's law of motion, we get

\[ m u'' = F_{\text{net}} = mg + F_s(t) + F_d(t) + F(t) \]
\[ = mg - k ( Lt u(t) ) - \gamma u' + F(t) \]

where \( F(t) \) is an external force.

Then, since \( mg = kl \), we get,

\[ m u'' + \gamma u' + ku = F(t) \]

where constants \( m, \gamma \), and \( k \) are positive.

Remark: These forces & the eqn we get is just an approximation. There are many other factors we don't take into account.

Example: A mass weighing 4kg stretches a spring 2 cm. Suppose that the mass is given an additional 6 cm displacement in the positive direction, then released. The mass is in a
medium that exerts a viscous resistance of 6 kg when the mass has velocity 3 m/s. Under the assumptions discussed above, formulate the initial value problem that governs the motion of the mass. (Assume $g = 10 \text{ m/s}^2$)

To write the eqn, we need to find $m$, $\gamma$, $h$, assuming that the external force is $F(t) = 0$.

For $m$, we see

$$mg = 4 \text{ kg} \Rightarrow m = \frac{4 \text{ kg}}{10 \text{ m/s}^2}$$

$$\Rightarrow m = \frac{2}{5} \text{ kg} \cdot \text{s/m}.$$

To find $\gamma$, we use

$$F_d = -\gamma u \Rightarrow 6 \text{ kg} = \gamma \cdot 3 \text{ m/s}$$

$$\Rightarrow \gamma = 2 \text{ kg} \cdot \text{s/m}.$$
To find $k$, we use the eqn

$$kL = mg \approx 4 \text{ kg} \implies k = \frac{4 \text{ kg}}{L \text{ meters}}$$

we see $L = 2 \text{ cm} = 0.02 \text{ m}$.

$$\implies k = \frac{4 \text{ kg}}{0.02 \text{ m}} = 200 \text{ kg/m}.$$

3) The eqn becomes

$$\frac{2}{5} u'' + 2u' + 200u = 0 \quad \text{or}$$

$$u'' + 5u' + 500u = 0,$$

with the initial conditions $u(0) = 0.06 \text{ m} \quad \& \quad u'(0) = 0 \quad \text{(no initial velocity)}$

**Undamped free vibrations**: This is the case when there is no damping force & no external force. Then we have the eqn
\[ m u'' + ku = 0, \text{ which has the characteristic eqn} \]

\[ mr^2 + k = 0 \implies r = \pm i \sqrt{\frac{k}{m}} \]

\[ \implies \text{The general soln of the eqn is} \]

\[ u = A \cos(w_0 t) + B \sin(w_0 t) \]

where \( w_0 = \sqrt{\frac{k}{m}} \).

Then, using trigonometric identities, we can write \( u \) as

\[ u = R \cos(w_0 t - \delta) \text{ where} \]

\[ A = R \cos \delta, \quad B = R \sin \delta \implies \]

\[ R = \sqrt{A^2 + B^2}, \quad \tan \delta = \frac{B}{A}. \]

This tells us that the motion of the mass is periodic (or simple harmonic). The period of the motion is
\[ T = \frac{2\pi}{w_0} = 2\pi \left( \frac{m}{k} \right)^{1/2}. \]

The circular frequency \( w_0 = \sqrt{\frac{k}{m}} \) measured in radians per unit time is called the natural frequency of the vibration. The max. displacement \( R \) is called the amplitude of the motion \( u \) & \( \delta \) is called the phase angle.

Ex. Suppose that a mass weighing 10 kg stretches a spring 20 cm. If the mass is displaced an additional 20 cm and is then
set in motion with an initial upward velocity $1 \text{ m/s}$, determine the position of the mass at any given time. (Take $g = 10 \text{ m/s}^2$)

We see that the spring constant, $k$, is

$$k = \frac{10 \text{ kg}}{0.20 \text{ m}} = 50 \text{ kg/m} \quad \Rightarrow \quad \text{the mass is}$$

$$m = \frac{F_c}{g} = \frac{10 \text{ kg}}{10 \text{ m/s}^2} = 1 \text{ kg} \cdot \text{s}^2/\text{m}.$$  

Then, the eqn $mu'' + ku = 0$, becomes

$$u'' + 50u = 0.$$  

$\Rightarrow$ The characteristic eqn for this eqn is

$$r^2 + 50 = 0 \quad \Rightarrow \quad r = \pm 5\sqrt{2} \text{i}$$

$\Rightarrow$ The general soln is

$$u = A \cos (5\sqrt{2}t) + B \sin (5\sqrt{2}t).$$  

where the initial conditions are
\[ w(0) = 0.2 \text{ m} \quad \text{and} \quad w'(0) = -1 \text{ m/s}. \]

\[ \Rightarrow A = \frac{1}{5} \quad \text{and} \quad B = \frac{-1}{5\sqrt{2}} = \frac{-\sqrt{2}}{10}. \]

\[ \Rightarrow u = \frac{1}{5} \cos(5\sqrt{2}t) - \frac{\sqrt{2}}{10} \sin(5\sqrt{2}t). \]

\[ \Rightarrow \text{The natural frequency } w_0 = 5\sqrt{2} \quad \text{and the period is} \quad T = \frac{2\pi}{w_0} = \frac{2\sqrt{2}\pi}{10}. \]

Then, to find the amplitude we find

\[ R^2 = A^2 + B^2 = \frac{1}{25} + \frac{1}{50} = \frac{3}{50}. \]

\[ \Rightarrow R = \frac{1}{5}\sqrt{\frac{3}{2}} \quad \text{and the phase angle} \]

\[ \delta \text{ satisfies} \quad \tan \delta = \frac{A}{B} = -\sqrt{2}. \quad \text{This eqn has two solns for } \delta, \quad \text{but since} \]

\[ R \cos \delta = A = \frac{1}{5} > 0 \quad \text{and} \quad R \sin \delta = \frac{-1}{5\sqrt{2}} < 0, \quad \text{we have one soln in the fourth quadrant}, \]
\[ \phi = \arctan(\sqrt{2}) \] & we get the graph