Introduction:

First, let's recall some definitions. Given an \( n \)-times differentiable function \( y \) that depends on one variable, say \( t \), the equation

\[
F(t, y, y', y'', \ldots, y^{(n)}) = 0
\]

where \( F \) is a multivariable func is called an ordinary differential equation. If \( y \) is a multivariable func \& the partial derivatives exist in the equation, then the eqn is called a partial differential equation. The highest order of derivative that appears
in the equation is called the order of a differential equation.

Example: \(y''' + 2e^t y'' + yy' = t^4\) is a third order differential equation, where \(y = y(t)\).

We call a differential equation

\[ L(y) = F(t, y, y', \ldots, y^{(n)}) = 0 \]

a linear equation if \(F\) is a linear func of the variables \(y, y', \ldots, y^{(n)}\). What this means is that if we replace \(y\) with \(y_1 ty_2\), and of course all the derivatives with then, then we get

\[ L(y_1 + ty_2) = L(y_1) + L(y_2). \]

And if we replace \(y\) with \(c \cdot y\) where \(c\) is a constant, and the corresponding derivatives accordingly, we get
\[ L(cy) = cL(y). \]

An eqn that is not linear is called a nonlinear equation.

Ex: \( y'' + 3ty' + e^t y = 0 \) is a linear equation since if we replace \( y \) with \( y_1 + ty_2 \), we get

\[
L(y_1 + ty_2) = (y_1' + ty_2')'' + 3t(y_1' + ty_2)' + e^t(y_1' + ty_2)
\]

\[
= (y_1'' + 3ty_1' + e^t y_1) + (y_2'' + 3ty_2' + e^t y_2)
\]

\[
= L(y_1) + L(y_2).
\]

Similarly

\[
L(cy) = (cy)'' + 3t(cy)' + e^t(cy)
\]

\[
= cy'' + c3ty' + ce^t y
\]

\[
= c((y'' + 3ty' + e^t y)) = cL(y)
\]

where \( c \) is a constant.
Thus the eqn \( y'' + 3ty' + e^t y = 0 \) is a linear eqn.

Ex: The eqn \( y'' y + y' = 0 \) is not linear since
\[
(y + y_2)'(y + y_2) + (y + y_2)' = (y'' + y_2')(y_1 + y_2) + (y_1 + y_2') \\
\neq (y'' + y_1 + y_1') + (y_1 + y_2 + y_2')
\]

We see that the general linear ordinary differential eqn of order \( n \) is

\[ a_0(t) y^{(n)} + a_1(t) y^{(n-1)} + \ldots + a_n(t) y = g(t) \]

A solution of the \( n \)th order ord. diff’l eqn

\[ F(t, y, y', \ldots, y^{(n)}) = 0 \]

on the interval \( \alpha < t < \beta \) is a func \( \phi \), such that \( \phi, \phi', \ldots, \phi^{(n)} \)
exist & satisfy

\[ F(t, \phi(t), \phi'(t), \ldots, \phi^{(n)}(t)) = 0 \]
Ex: Consider the eqn \( y'' + ty = 0 \).

We see that this is a second order linear homogeneous differential equation and the function \( y_1 = \cos t \) is a solution since \( y_1' \) and \( y_1'' \) exist and moreover \( y_1' = -\sin t \) and \( y_1'' = -\cos t \), which means

\[ y_1'' + ty_1 = -\cos t + \cos t = 0. \]

We can also check that \( y_2 = \sin t \) is also a solution.

First Order Differential Equations

Chapter 2.1: Linear Differential Equations, Integrating Factors

In this chapter we are going to learn the method of integrating factors to solve a general first order linear differential equation in
the standard form

\[(*) \quad \frac{dy}{dt} + p(t)y = g(t) \quad \text{where} \quad p(t) \& g(t) \quad \text{given fncs of the variable} \ t, \ \text{or} \ \text{of the form} \]

\[(**) \quad P(t) \frac{dy}{dt} + Q(t)y = G(t), \quad \text{where} \ P, Q \ & G \ \text{are given fncs.} \ \text{We can easily see that if} \ P(t) \to 0, \ \text{we can write} \ (** \ as \ an \ eqn \ of \ the \ form} \ (* \ by \ dividing \ the \ eqn \ by \ P(t).)

\text{To understand the method, we need to recall the product rule of differentiation:}

\text{If we have two diff'f fncs} \ \mu \ & y, \ \text{then} \ (\mu y)' = \mu' y + \mu y' \quad \text{----} \ (**)

\text{But if we look at the eqn} \ (*), \ y' + p(t)y = g(t) \ \text{we find}:

\[ (**') \quad \frac{dy}{dt} + p(t)y = g(t) \]
we see that the left hand side (LHS) of this eqn resembles \((\star\star)\), but not quite the same. We see however that if we multiply the eqn \((\star)\) by \(\mu(t)\), we get
\[
\mu(t) y' + \mu(t) p(t)y = \mu(t) g(t). \quad (4)
\]
This tells me that if we have
\[
\mu(t) p(t) = \mu'(t),
\]
then the left hand side of \((4)\) becomes a full derivative, \(\mu y'\).

Now, first let's find out what that \(\mu\) should be. If we assume temporarily that \(\mu(t)\) is positive, dividing \(\mu(t) p(t) = \mu'(t)\) by \(\mu(t)\), we get
\[
p(t) = \frac{\mu'(t)}{\mu(t)}.\]

Nice thing about this eqn is that we see that the right hand side (RHS) is a full derivative, namely
\[
\frac{\mu'(t)}{\mu(t)} = (\ln(1/w))' = p(t).\]
Thus, integrating
both sides, we get
\[ \ln(1/n(\tau)) = \int p(t)dt + k \]
\[ i.e \quad n(t) = \exp(\int p(t)dt) \]
But then, with this \( n \), the eqn (4) becomes
\[ n'y = n(t)g(t) \] & hence
by integrating both sides we get
\[ n(t)y(t) = \int n(t)g(t)dt + c \] & hence
\[ y(t) = \frac{1}{n(t)} \left( \int_{t_0}^{t} n(s)g(s)ds + c \right) \]
where \( t_0 \) is some convenient lower limit of integration.

This may look a little intimidating, but in reality, the method is quite straightforward & if we remember where the formulas come from, solving these eqns will be easy.
Now, let's see it in an example.

**Ex:** Solve the initial value problem

\[
\begin{bmatrix}
ty' + 2y &= 4t^2 \\
y(1) &= 2
\end{bmatrix}
\]

To solve this eqn, let's put it in the standard form \((x)\) first, by dividing it by \(t\). Thus, we get

\((5)\) \[y' + \frac{2}{t} y = 4t^3\]. Here, we see that in general form, \(p(t) = \frac{2}{t}\) & \(g(t) = 4t^3\).

Now, first thing we need to do is to compute the integrating factor \(\mu(t)\):

\[
\mu(t) = \exp\left(\int \frac{2}{t} \, dt\right) = e^{\ln(t^2)} = e^{\ln(t^2)} = t^2.
\]

Then, multiplying the eqn \((5)\) by \(\mu(t) = t^2\), we get \(t^2 y' + 2t y = 4t^3\). \[---- (6)\]
(But then, from the previous arguments we had, we see that the LHS of (6) must be the derivative of \( \mu(t)y = t^2y \), and indeed \((t^2y)' = t^2y' + 2ty\).)

Hence we get
\[
(t^2y)' = 4t^3
\]
which implies
\[
t^2y = \int 4t^3 \, dt = t^4 + C,
\]
where \(C\) is an arbitrary constant. Thus
\[
y = \frac{1}{t^2}(t^4 + C)
\]
i.e.
\[
y(t) = t^2 + \frac{C}{t^2}.
\]
Although \(y(t)\) satisfy the eqn itself, it also has to satisfy the boundary condition
\[
y(1) = 2.
\]
This means that
\[
y(1) = 1^2 + \frac{C}{1^2} = 1 + C = 2.
\]
Thus \(C = 1\).

Therefore the solution is \(y(t) = t^2 + \frac{1}{t^2}, t > 0\).
Remark: We see that this solution fails to exist for all times and because of the infinite discontinuity in p(t) at t=0, which restricts the solution to the interval t>0 (or t<0, but since the initial data is given at t=1, we take the interval t>0)

Ex: Solve the initial value problem

\[
\begin{align*}
2y' + ty &= 2 \\
y(0) &= 1
\end{align*}
\]

In this question, to be able to apply the method we have seen above, we need to write the eqn in the standard form first, by dividing the eqn by 2 & thus making the coefficient of y', 1. Then we have the eqn
\[
\begin{align*}
\begin{cases}
y' + \frac{t}{2} y &= 1 \\
y(0) &= 1
\end{cases}
\end{align*}
\]

Thus, we see that \( p(t) = \frac{t}{2} \) & \( q(t) = 1 \), as given in the formula.

Then, we first find the integrating factor \( \mu(t) = \exp\left(\int \frac{t}{2} \, dt\right) = e^{t^2/4} \).

Then, multiplying the eqn in standard form by \( \mu(t) = e^{t^2/4} \) we get

\[
\begin{align*}
e^{t^2/4} y' + \left( e^{t^2/4} \cdot \frac{t}{2} \right) y &= e^{t^2/4} \\
\left( e^{t^2/4} y \right)' &= e^{t^2/4} 
\end{align*}
\]

\( \Rightarrow \left( e^{t^2/4} y \right)' = e^{t^2/4} \). Thus, integrating both sides, we get, by fundamental theorem of calculus,

\[
e^{t^2/4} y = \int e^{t^2/4} \, dt + C, \text{ where } C \text{ is an}
\]
arbitrary constant. Since the initial condition is given at \( t=0 \), and that the function \( e^{t^2/4} \) doesn't have an explicit antiderivative, we can write the eqn above as

\[
e^{t^2/4} y = \int_0^t e^{s^2/4} \, ds + c \quad \text{again, } c
\]

is an arbitrary constant. Then we see

\[
y(t) = e^{-t^2/4} \int_0^t e^{s^2/4} \, ds + ce^{-t^2/4},
\]

and since \( y(0) = 1 \), we get

\[
l = y(0) = e^0 \int_0^0 e^{-s^2/4} \, ds + ce^0
\]

\[
= 1 \overbrace{\int_0^0 e^{-s^2/4} \, ds + c}^{1} = 1 \quad (\text{integral of a continuous func from 0 to 0})
\]

\[
\Rightarrow \quad l = c.
\]

Therefore \( y(t) = e^{-t^2/4} \int_0^t e^{s^2/4} \, ds + e^{-t^2/4} \).
Separable eqns: We have seen that a general 1st order differential eqn can be written as (in the independent variable $x$)

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0. \quad \text{(5)}$$

In this section, we are going to take a look at a specific case of such eqns where $M$ doesn't depend on $y$ & $N$ doesn't depend on $x$, i.e. $M(x, y) = M(x)$ & $N(x, y) = N(y)$.

Then the eqn (5) turns into

$$M(x) + N(y) \frac{dy}{dx} = 0. \quad \text{(6)}$$

Such an equation is called separable, because if it is written in differential form

$$M(x) \, dx + N(y) \, dy = 0,$$

we can separate the $x$ & $y$ variable terms into opposite sides of the eqn.
To solve the eqns of the form (6), assume that \( H_1(x) \) & \( H_2(y) \) are antiderivatives of \( M(x) \) & \( N(y) \) respectively, that is

\[
M(x) = \frac{d}{dx} H_1(x) \quad \text{&} \quad N(y) = \frac{d}{dy} H_2(y).
\]

Then we see that the eqn (6) becomes

\[
\frac{d}{dx} H_1(x) + \frac{d}{dy} H_2(y) \frac{dy}{dx} = 0.
\]

This is just \( \frac{d}{dx} H_2(y) \) using chain rule.

\[1) \quad \frac{d}{dx} H_1(x) + \frac{d}{dx} (H_2(y)) = 0.\]

\[2) \quad \frac{d}{dx} \left( H_1(x) + H_2(y) \right) = 0, \text{ which means }\]

\[H_1(x) + H_2(y) = C \quad \text{where } c \text{ is a constant.}\]

Moreover, if the initial condition \( y(0)=y_0 \),
is given, we see
\[ c = H_1(x_0) + H_2(y_0) \]

Then, putting this \(c\) in the eqn above, and noticing that
\[ H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s) \, ds \]
\[ H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s) \, ds \]
(by Fundamental thm of Calc.)

we get
\[ \int_{x_0}^{x} M(s) \, ds + \int_{y_0}^{y} N(s) \, ds = 0. \]

Heuristically what this tells us is that when we write eqn (6) in the form
\[ M(x) \, dx + N(y) \, dy = 0, \]
we can integrate both sides to get
\[ \int M(x) \, dx + \int N(y) \, dy = 0. \]

Now, let's see this in one example.
Ex: Solve the initial value problem
\[
\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1.
\]

& determine the interval in which the soln exists.

We see that this eqn can be written as
\[
2(y-1) \frac{dy}{dx} = (3x^2 + 4x + 2) \quad \text{or as}
\]
\[
2(y-1) dy = (3x^2 + 4x + 2) dx.
\]
Then, by the arguments above, integrating both sides we get (LHS in \( y \), RHS in \( x \))
\[
y^2 - 2y = x^3 + 2x^2 + 2x + C \quad \text{where \( C \) is an arbitrary constant.}
\]

Then, checking the initial condition, we get
\[
y^2(0) - 2y(0) = 0^3 + 2(0^2) + 2(0) + C \quad \text{and since}
\]
\[ y(x) = -1, \text{ we get} \]
\[ (-1)^2 - 2(-1) = c \implies c = 3. \]

Therefore the soln \( y \) satisfies the eqn
\[ y^2 - 2y = x^3 + 2x^2 + 2x + 3 \]

Completing the LHS into a complete square, we get
\[ y^2 - 2y + 1 = (y - 1)^2 = x^3 + 2x^2 + 2x + 4 \]
\[ = y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4} \]

& since \( y(0) = -1 \), we see that we need to take the "-" sign i.e
\[ y(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \]

To find the interval on which the soln is valid we need to find the interval on which the expression inside square-root is positive. The only real root of the
expression \( x^3 + 2x^2 + 2x + 4 \) is \( x = -2 \) & for \( x > -2 \), \( y(x) \) is well defined. Therefore the soln is \( y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \) on \( x > -2 \).

You can see the integral curves of the eqn \( y' = \frac{3x^2 + 4x + 2}{2(y - 1)} \).
Ex: Solve the separable differential eqn
\[
\frac{dy}{dx} = \frac{4x-x^3}{4+xy^3}
\]
and draw graphs of several integral curves.
Find the soln that passes through the point (0,1) & find its interval of existence.

First let's write the eqn in differential form as
\[(4+xy^3)dy = (4x-x^3)dx.
\]
Then, integrating both sides, we get
\[
4y + \frac{y^4}{4} = 2x^2 - \frac{x^4}{4} + c, \text{ where } c \in \mathbb{R}
\]

Now, if we multiply both sides by 4 & collecting x & y terms together, we get
\[
y^4 + 16y + x^4 - 8x^2 = c \quad c \in \mathbb{R}.
\]
Unfortunately there is no easy way to solve a 4th order polynomial (since any polynomial of order greater than 3 doesn’t have radical-soln-formula like quadratic formula). So we leave the soln in implicit form. If we check the initial condition $y(0)=1$, we see

\[ y(0) + 16y(0) + 0 - 0 = C \Rightarrow 1 + 16 = C \Rightarrow C = 17. \]

Thus, the soln that passes through $(0,1)$ is

\[ y + 16y + x^4 - 8x^2 = 17. \]

Now, if we look at the slope field & the integral curves of this eqn, we see the curve that passes thru $(0,1)$ fails to be a fnce when tangent line is vertical, i.e. when

\[ \frac{dy}{dx} = \frac{4x-x^3}{4+y^2} \] goes to infinity. That is,
when \( 4 + y^3 = 0 \) \( \Rightarrow y = -\sqrt[3]{4} \) & the \( x \) pts that correspond to this value of \( y \) are \( x \approx \pm 3.3488 \).

Chapter 2.3: Modeling with 1st order diff. eqns.

Differential eqns, ordinary or partial, are one of the main tools used to understand physical phenomena. The process of expressing a physical, biological, chemical etc. phenomena
in mathematical terms is called mathematical modeling. There are 3 main steps in modeling.

**Step 1:** Construction of the model

**Step 2:** Analysis of the model where we try to solve or try to understand the differential eqns in the model.

**Step 3:** Comparison with Experiment or observation where we look at the soln & compare it to experimental or observational data.

Ex: At time \( t=0 \) a tank contains \( Q_0 \) lb of salt water dissolved in 100 gal of water. Assume that water containing \( \frac{1}{4} \) lb of salt per gallon is entering the
tank at a rate of \( r \text{ gal/min} \), and that the well-stirred is draining from the tank at the same rate. Set up the initial value problem that describes the flow process. Find the amount of salt \( Q(t) \) in the tank at any given time, also find the limiting amount \( Q_L \) that is present after a long time. If \( r = 3 \text{ gal/min} \) and \( Q_0 = 2Q_L \), find the time \( T \) after which the salt level is within \%2 of \( Q_L \). Also find the flow rate that is required if the value of \( T \) is not to exceed 45 min.

\( r \text{ gal/min}, 1/4 \text{ lb/gal} \)
To set up the initial value problem all we need to observe that the change in salt levels is due to the salt flowing in & flowing out & thus the rate of change of the salt amount is
\[ \frac{dQ}{dt} = (\text{rate of salt that goes in}) - (\text{rate of salt that goes out}) \]

Since we know that the flow of the salt water pumping in is constant $r$ gal/min & with concentration $\frac{1}{4}$ lb/gal, we see the rate of change of the salt amount that goes in is $r \cdot \frac{1}{4} = \frac{r}{4}$ lb/min.

Now, let's calculate the rate of change of amount of salt that goes out at any time $t$. For that we need to do a similar argument as above & we will calculate
Rate out = (Concentration at time $t$) $\cdot$ (flow rate)  
= (Concentration at time $t$) $\cdot$ $r$

But now, the question is, how can we calculate the concentration of salt in the tank at any given time? This is not too difficult, all we remember is that Concentration = $\frac{\text{Amount}}{\text{Volume}}$ and we know amount of salt at any given time is represented as $Q(t)$ & volume of the tank is 100 gallons. Thus, concentration at any given time is $\frac{Q(t)}{100}$. Therefore the Rate out $= r \cdot \frac{Q(t)}{100}$. This gives us the differential equation

$$\frac{dQ}{dt} = -\frac{r}{4} \cdot \frac{Q(t)}{100}, \quad \text{with } Q(0) = Q_0$$
Observations: If we keep pouring in a salt mixture of concentration \( \frac{1}{4} \) lb/gal, we would expect the salt level in the tank to converge to the same concentration level, \( \frac{1}{4} \) lb/gal, i.e., amount of salt should converge to \( \frac{1}{4} \cdot 100 = 25 \) lbs.

We also see that at this limiting value, the amount of salt shouldn't change. This means that if we set \( \frac{dQ}{dt} = 0 \), we should get the limiting amount of salt, i.e.,

\[
0 = \frac{dQ}{dt} = \frac{r}{4} - \frac{Q}{100} \Rightarrow Q = 25 \text{ lb}.
\]

Now, let's solve this differential eqn,

\[
\frac{dQ}{dt} + \frac{Q}{100} = \frac{r}{4}.
\]
We see that this is a first order linear ODE, written in the standard form. For this eqn, we can use the integrating factor, where \( p(t) = \frac{r}{100} \) & \( q(t) = \frac{r}{4} \).

Then \( \mu(t) = \exp \left( \int \frac{r}{100} \, dt \right) = e^{\frac{rt}{100}}. \) Then multiplying the eqn (which is already in std. form) by the integrating factor, we get

\[
\frac{d}{dt} \left( e^{\frac{rt}{100}} Q(t) \right) = \frac{r}{4} e^{\frac{rt}{100}}.
\]

Then, by integrating,

we get

\[
e^{\frac{rt}{100}} Q(t) = \int \frac{r}{4} e^{\frac{rt}{100}} \, dt
\]

\[
e^{\frac{rt}{100}} Q(t) = 25 e^{\frac{rt}{100}} + C
\]

\[
\Rightarrow Q(t) = \frac{(25 e^{\frac{rt}{100}} + C)}{e^{\frac{rt}{100}}} = 25 + C e^{-\frac{rt}{100}}
\]
where \(c \leq 12\). Then, given the initial condition \(Q(0) = Q_0\), we see \(c = Q_0 - 25\).

\[ Q(t) = 25 + (Q_0 - 25) e^{-\frac{r}{100}t} \]

Thus, we see that

\[ \lim_{t \to \infty} Q(t) = 25. \]

Now, for the second part of the problem, let \(r = 3\) \& \(Q_0 = 2Q_L = 50\). Then the eqn for \(Q\) becomes

\[ Q(t) = 25 + (50 - 25) e^{-\frac{3}{100}t} \]

\[ = 25 + 25 e^{-0.03t}. \]

Now we want this to be within 2\% of 25 \(i.e.,\) we want the value \(Q(t)\) to be between \((24.5, 25.5)\). But, since \(Q_0 = 50\), we expect to get \(Q(t) < 25.5\).
If we substitute $Q(t_0) = 25.5$ in the eqn for $Q(t)$, we get

$$25.5 = 25 + 25e^{-0.03t_0}$$

$$\Rightarrow 0.5 = 25e^{-0.03t_0}$$

$$\Rightarrow \frac{1}{50} = e^{-0.03t_0}$$

$$\Rightarrow 50 = e^{0.03t_0}$$

$$\Rightarrow t_0 = \frac{\ln(50)}{0.03} \approx 130.4 \text{ mins.}$$

If we want to calculate the rate $r$ so that we get $2\%$ of $Q_e$ in $T = 45$ mins, we need to plug the values in the eqn

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100}$$

where $Q(t) = 25.5$, $Q_0 = 50$, $T = 45$.

Then, we get

$$25.5 = 25 + (25)e^{-0.45r}$$

$$\Rightarrow r = \frac{100}{45} \ln 50 \approx 8.69 \text{ gal/min}.$$
Ex: A body of constant mass \( m \) is projected away from the earth in a direction perpendicular to the earth’s surface with the initial velocity \( v_0 \). Assuming that there is no air resistance, but taking into account earth’s gravitational field with distance, find the expression for the velocity during the ensuing motion. Also find the initial velocity that is required to lift the body to a given maximum altitude \( A_{max} \) above the surface of the earth and find the least velocity for which the body will not return to the earth (which is called the escape velocity).
To find the differential eqn of the velocity, we are going to use the eqn of motion, 

\[ m \cdot a = F. \]

To be able to do that, we need to find the forces acting on the object, which in this case, is the gravitational force.

Recall that the gravitational force

\[ W(x) = -\frac{m \cdot M \cdot G}{(x+R)^2} \]

where \( M \) is the mass of the Earth, \( G \) is the gravitational constant.
Thus, we also see that the numerator is constant, and we know at $x=0$, the force $w(x) = -mg$. Thus, we see that

$$-mg = \frac{-mgR}{(0+R)^2} \Rightarrow MG = gR^2, \text{ i.e.}$$

$$w(x) = -\frac{mgR^2}{(x+R)^2}.$$  

Therefore, since gravitational force is the only force acting on the object, we get

$$ma = m\frac{dv}{dt} = \frac{-mgR^2}{(R+x)^2} \text{ with the initial condition } v(0) = v_0. \text{ Now, we see that we have two independent variables } t \text{ and } x. \text{ But we can solve this problem using chain rule:}$$

we see

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dt} \frac{1}{v}.$$
Therefore, the eqn becomes

\[ m \frac{dv}{dt} = m \frac{dv}{dx} \cdot \sqrt{v} = -\frac{mgR^2}{(R+x)^2} \]

\[ \Rightarrow \frac{dv}{dx} \sqrt{v} = -\frac{gR^2}{(R+x)^2} \quad (\text{where we see that we write } v \text{ as an eqn in } x) \]

We see that this eqn is separable. If we write it in differential form, we get

\[ v \, dv = -\frac{gR^2}{(R+x)^2} \, dx \]

\[ \text{& integrating both sides we get,} \]

\[ \frac{v^2}{2} = \frac{gR^2}{(R+x)} + C. \]

Now, since at \( t=0 \), we have \( x=0 \), we get

\[ \frac{v(0)^2}{2} = \frac{gR^2}{R} + C \quad \Rightarrow \quad C = \frac{v_0^2}{2} - \frac{gR^2}{R} \]

\[ \Rightarrow \quad v = \sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}, \quad \text{now, since} \]

\[ \text{...} \]
the object is moving in $+x$-direction, we need to take the "+" sign above i.e.

$$v = \sqrt{v_0^2 - 2gR + \frac{2gR^2}{R + x}}.$$ 

To determine the maximum altitude $A_{\text{max}}$, we need to set $x = A_{\text{max}}$ & since it is the max distance, we set $v = 0$.

$$A_{\text{max}} = \frac{v_0^2 R}{2gR - v_0^2}.$$ 

Thus, if we solve for $v_0$, we get

$$v_0 = \sqrt{2gR} \frac{A_{\text{max}}}{R + A_{\text{max}}}.$$ 

And we can find the escape velocity, $v_e$, by sending $A_{\text{max}}$ to $\infty$, i.e.

$$v_e = \lim_{A_{\text{max}} \to \infty} v_0 = \sqrt{2gR} \approx 11.1 \text{km/s}.$$
Chapter 2.4: Differences Between Linear & Nonlinear eqns:

- Existence & uniqueness:
  
  Theorem 2.4.1: (Existence & uniqueness for first order linear eqns)

  If the funs $p$ & $g$ are cts on an open interval $I: a < t < b$ containing the point $t = t_0$, then there exists a unique soln $y = f(t)$ to the eqn $y' + p(t)y = g(t)$ for each $t \in I$, and also satisfies the initial condition $y(t_0) = y_0$ where $y_0 \in \mathbb{R}$.

  (We have seen how to solve such eqns, where the soln depends on the integrability of $p(t)$ (to find $\mu(t)$, the integrating factor) & integrability of $(\mu, g)$. The conditions in the
Then guarantees the existence of all these expressions. 

Whereas for the nonlinear eqns, the thm is fundamentally different.

**Thm 2.4.2: Existence & Uniqueness Thm for 1st order nonlinear diff’l eqns.**

Let the funcs \( f \) & \( \partial f/\partial y \) be cts in some rectangle \( \alpha < t < \beta, \gamma < y < \delta \) containing the point \((t_0, y_0)\). Then, in some interval \(-h < t < h + t_{toth}\) contained in \( \alpha < t < \beta \), there is a unique soln \( y = \phi(t) \) of the initial value problem

\[
y' = f(t,y), \quad y(t_0) = y_0.
\]

---

• We see that if we apply this thm to linear eqns \( y' = -p(t)y + g(t) \), we see
that \( f(t, y) = -p(t)y + g(t) \) \& \( \frac{df}{dy}(t, y) = -p(t) \)
being cts is equivalent to \( p \) & \( g \) being cts (continuous). But by the light of Thm 2.4.1, we don't need the special subinterval \((t_0 - h, t_0 + h)\).

Ex: Consider the eqn

\[
\begin{bmatrix}
y' + 2y = 4t^2 \\
y(1) = 2
\end{bmatrix}
\]

We see that this is a 1st order linear eqn, which can be written in standard form as

\[
y'(\frac{2}{t})y = 4t. \text{ So } p(t) = \frac{2}{t} \text{ & } g(t) = 4t.
\]

We see that \( g(t) \) is a cts func whereas \( p(t) \) is cts on \((0, \infty) \) \& \((-\infty, 0)\). Then given the initial condition \( y(1) = 2 \), by Thm 2.4.1, we would expect to have a unique soln
for \( 0 < t \). Indeed, we solved this eqn & got 
\[ y(t) = t^2 + \frac{1}{t^2} \], which exists on \( t > 0 \).

**Ex:** Consider the initial value problem
\[
\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1.
\]

In light of Thm 2.4.2, we see that
\[
f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)} \quad \text{&} \quad \frac{\partial f(x,y)}{\partial y} = \frac{(3x^2 + 4x + 2)}{2(y-1)^2}.
\]

Both of these fgs are cts everywhere except for when \( y = 1 \). Therefore, we can find a unique soln in a rectangle around \((0, -1)\) (the initial condition).

Indeed, we solved this eqn & got
\[ y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \] which exists for all \( x > -2 \).
Moreover, if we change the initial cond. to \( y(0) = 1 \) (recall that this is where \( f(x,y) \) \& \( \frac{\partial f}{\partial y}(x,y) \) is discontinuous).

Then, solving the eqn again with this initial data, we get

\[
y = 1 + \sqrt[3]{x^3 + 2x^2 + 2x}
\]

which is not unique (two solns, one for "+" one for ")

But this is not a problem since the Thm 24.1 didn't predict this soln to be unique when \( y = 1 \).

**Ex:** Consider the initial value problem

\[
y' = y^{1/3}, \quad y(0) = 0.
\]

In this question, we see

\[
f(t,y) = y^{1/3} \text{ which is a cts fnc but } \frac{\partial f}{\partial y}(t,y) = \frac{1}{3} y^{-2/3} \text{ does not exist when } y = 0.
\]
This means that, with the initial value given, Thm 2.4.2 doesn't apply. Indeed, this is a separable eqn & we can solve it quite easily (without focusing on the initial data).

We can write the eqn in differential form as

\[ y^{-\frac{1}{3}} \, dy = dt \]

& integrating both sides we get

\[ \frac{3}{2} y^{\frac{2}{3}} = t + c \implies y = \left( \frac{2}{3} (t+c) \right)^{\frac{3}{2}} \]

& checking the initial condition \( y(0) = 0 \), we get \( c = 0 \) \( \implies y = \left( \frac{2}{3} t \right)^{\frac{3}{2}} \).

But here, we also see that \( y = 0 \), the constant 0 fnc is also a soln to this eqn & also satisfies the initial cond.
This means that the solution is not unique. Moreover, there are infinitely many solutions of the form

$$y = \begin{cases} 
0 & t < t_0 \\
\left(\frac{2}{3}(t-t_0)^{3/2} & t \geq t_0 
\end{cases}$$

for any $t_0 > 0$.

**Ex:** Solve the initial value problem

$$y' = y^2, \quad y(0) = 1$$

and determine the interval in which the solution exists.

First, we see that $f(t, y) = y^2$ and $\frac{\partial f}{\partial y} = 2y$ which are both continuous everywhere.

So, we are in the setup of Thm 24.2, which means that there has to be a unique solution in a rectangle around $(0, 1)$.
Now, let's solve this eqn. We see that this is a separable eqn & in differential form, it is

\[ y^{-2}\,dy = dt \quad \text{\& integrating both sides, we get} \]

\[ -y^{-1} = t + c \quad \text{i.e.} \quad y = \frac{-1}{t + c} \]

Checking the initial cond. \( y(0) = 1 \), we get
\[ c = -1 \quad \text{i.e.} \quad y = \frac{-1}{t - 1} = \frac{1}{1 - t} \]

This tells us that the solution goes to \( \infty \) as \( t \to 1 \). Thus, the solution exists in \( -\infty < t < 1 \), which we couldn't see from the eqn itself & the conditions on Thm 24.2.

This tells us that when dealing with differential eqns, the linear eqns are much
nicer & more predictable than nonlinear diff. eqns.

Chapter 2.5: Autonomous Differential Eqns

In this chapter we are going to consider the eqns of the form

\[
\frac{dy}{dt} = f(y), \quad i.e. \text{ where the }
\]

eqn doesn't depend explicitly on the independent variable. There are two important examples of such eqns:

1. **Exponential growth:** let \( y = \phi(t) \) be the population of a given species at time \( t \).
   If we assume that the rate of change of the population \( y \), is proportional to the current value of \( y \), that is
\[
\frac{dy}{dx} = r \cdot y \quad \text{where the proportionality constant } r \quad \text{is called rate of growth or decline.}
\]
If \( r > 0 \), then the population is growing & if \( r < 0 \), then the population is declining.

If we assume that \( r > 0 \), we can solve this eqn either by integrating factor or separation. Say, we use separability of the eqn & write it as

\[
\frac{dy}{y} = r \, dt.
\]
Then integrating both sides we get

\[
l \ln |y| = rt + c \quad \text{or} \quad e^{rt} \ln |y| = c.
\]

\[
\Rightarrow y = C e^{rt}. \quad C \in \mathbb{R}.
\]

Then, if the initial value \( y(0) = y_0 \) is given, we get \( C = y_0 \) i.e

\[
y = y_0 e^{rt}.
\]
which says that the population grows exponentially. (We can see this as a model for bacterial growth with sufficient food and space, where every cell doubles in certain time, i.e., \( r = \ln 2 \))

2) Logistic growth: In real life, the exponential growth is practically impossible since the population also affects the amount of food & space available & affects the growth. To take this into account we can replace the proportionality constant with a function of the population, \( y \), i.e. we get

\[
\frac{dy}{dt} = h(y) \cdot y.
\]

One of the most observed version of this
eqn is when \( h(y) = r > 0 \) when \( y \) is small i.e. the population grows almost exponentially & \( h(y) \) decreases as \( y \) increases & \( h(y) < 0 \) for \( y \) sufficiently large i.e. population declines if it is too big (especially for the environment to support it).

Simplest such \( h(y) \) is \( h(y) = r - ay \)
where \( a > 0 \). Using this func. we get

\[
\frac{dy}{dt} = (r - ay)y.
\]

This eqn is called the logistic eqn & can also be written as

\[
\frac{dy}{dt} = r (1 - \frac{y}{k})y \quad \text{where } k = \frac{r}{a}.
\]

In this form, \( r \) is called the intrinsic growth rate. We see that this eqn has
two constant solns $y = \Phi_1(t) = 0$ & $y = \Phi_2(t) = k$ which are called the equilibrium solns since they correspond to no change in population. These solns can also be found by locating the roots of $f(y)$, i.e.,

$$\frac{dy}{dt} = f(y) = 0$$. The zeroes of $f(y)$

are also called critical pts.