Difference Eqns:

- Special type of recursive relations

\[ X_{t+1} - X_t = F(X_t) \]

\[ X_t: \ \text{present state} \]
\[ X_{t+1}: \ \text{next state} \]

e.g.:

\[ X_{t+1} - X_t = r X_t \]

\[ a_{n+1} - a_n = \frac{1}{2} a_n^2 \]

Here, \( t, n \) denote the discrete time steps.

e.g. # of generations
# of hours
# of months, etc.

\[ X_0, \ X_1 = (1+r) X_0 \Rightarrow X_2 = (1+r) X_1 = (1+r)^2 X_0 \]
\[ 3 = (1+r) X_2 = (1+r)^3 X_0 \]

\[ \Rightarrow X_t = (1+r)^t X_0 \]

We see using first linear approx, we can approximate

\[ X_{t+1} - X_t \approx \frac{dx}{dt} \Delta t \]

discrete time difference.
This suggests

\[ x_{t+1} - x_t = rF(x_t) \Rightarrow \frac{dx}{dt} = F(x_t) \]

are somewhat related if \( r \) is small, where \( r \) can be seen as \( \Delta t \). We also see this approx is better if \( \Delta t \) is small (i.e. \( r \) is small).

Thus, if \( r \) is big enough they have have different behaviours.

**Logistic Map:**

\( N_t \) = population at time step \( t \).

\[ N_{t+1} = aN_t - bN_t^2 \]

We can see this as

\[ N_{t+1} - N_t = (a-1)N_t - bN_t^2 \] (similar to \( \frac{dN}{ds} = rN - \frac{r}{k}N^2 \))

\( \Rightarrow \) We can see the logistic map as

\[ N_{t+1} = g(N_t) \]

\( g(N) = aN - bN^2 = N(a-bN) \)

\( \Rightarrow \) The zeros of \( g \) are \( N = 0 \)

\& \( N = \frac{a}{b} \).
We see $y = N^*$.

\[\Rightarrow\text{ fixed pts are } N = 0 \quad \& \quad N = N^*.\]

What happens when $N_t = N^*$? $N_{t+1} = G(N_t) = N_t = N^*$.

What about $N_t < N^*$

\[\Rightarrow N_t < N^* \Rightarrow N_{t+1} > N_t.\]

If $N_t > N^*$, then $N_{t+1} < N_t$. 
For simplicity, take
\[ x_{t+1} = g(x_t), \quad g(x) = \alpha x (1-x), \quad (x > 0). \]
(by simply taking
\[ x_t = \frac{x}{N_t}. \])

The behaviour of the logistic map
\[ x_{t+1} = g(x_t), \quad g(x) = \alpha x (1-x) \]
depends on the values of \( \alpha \).

To understand this, first find fix pts.

these are pts. where \( x = g(x) = \alpha x (1-x) \).

\( \Rightarrow \) \( x = 0 \) is a fixed pt. or \( \Rightarrow \) \( \frac{1}{x} = 1 - x \).

\( \Rightarrow \) \( \frac{1}{\alpha} = 1 - x \).

The pt \( x^* = x = \frac{\alpha - 1}{\alpha} \) is called the carrying capacity.

We see for \( 0 < \alpha < 1 \), we get \( x^* = 1 - \frac{1}{\alpha} < 0 \) which is not physically possible, since \( x > 0 \) (population).

Thus, the only meaningful fixed pt. is \( x = 0 \).

Now, let's take a look;
Case: $0 < x < 1$

$y = x$ gets closer to 0.

$$\lim_{t \to \infty} x_t = 0.$$  

$y = g(x).$

Case: $1 < x$: Then we have the fixed pts.

$x = 0, \quad x^* = 1 - \frac{1}{x}$.

Now, we need to check the stability of fixed pts.

[We know if $V$ is a fixed pt, $|g'(V)| < 1$,

then $V$ is stable, if $|g'(V)| > 1$, unstable]
Now, let's check $g'(x)$.

$$g(x) = xx(1-x) = ax - ax^2$$

$$\Rightarrow g'(x) = x - 2ax.$$  

If $x > 1$, we have fixed pts $x = 0$ and $x^* = 1 - \frac{1}{a}$.  

At $x = 0$, we get $g'(0) = x > 1$. $\Rightarrow$ $x = 0$ is unstable.  

At $x^* = 1 - \frac{1}{a}$, $g'(x^*)$:  

$$g'(1 - \frac{1}{x}) = a - 2x(1 - \frac{1}{x}) = a - 2a + 2 = 2 - a.$$  

$\Rightarrow \left| g'(x) \right| < 1 \iff -1 < 2 - x < 1 \iff 1 < x < 3.$  

$\Rightarrow x^*$ is stable if $1 < x < 3$  

$x^*$ is unstable if $a > 3$.  