Another way of calculating limits:

1. If \( \lim_{n \to \infty} a_n = L \) and \( F(x) \) is cts at \( x = L \).

Then \( \lim_{n \to \infty} F(a_n) = F(L) \).

Example:

\[
\lim_{n \to \infty} \sin \left( \frac{\pi}{2} + \frac{(-1)^n}{n^2} \right).
\]

As we see sin(x) is cts.

we see \( \lim_{n \to \infty} \sin \left( \frac{\pi}{2} + \frac{(-1)^n}{n^2} \right) = \sin \left( \lim_{n \to \infty} \left( \frac{\pi}{2} + \frac{(-1)^n}{n^2} \right) \right) \).

we also see \( \lim_{n \to \infty} \left( \frac{\pi}{2} + \frac{(-1)^n}{n^2} \right) = \frac{\pi}{2} \). sinu

\[
\frac{\pi}{2} - \frac{1}{n^2} \leq \frac{\pi}{2} + \frac{(-1)^n}{n^2} \leq \frac{\pi}{2} + 1 \frac{1}{n^2}
\]

\[
\int_{n \to \infty}^{\frac{\pi}{2}} \int_{n \to \infty}^{\frac{\pi}{2}}
\]

\[
\lim_{n \to \infty} \sin \left( \frac{\pi}{2} + \frac{(-1)^n}{n^2} \right) = \sin \left( \lim_{n \to \infty} \frac{\pi}{2} + \frac{(-1)^n}{n^2} \right) = \sin \left( \frac{\pi}{2} \right) = 1.
\]

Example:

\[
\lim_{n \to \infty} n^n = ?
\]
We let \( a_n = n^{1/n} \). Now \( \ln(a_n) = \ln(n^{1/n}) \)

a let \( b_n = \ln(a_n) = \frac{1}{n} \ln(n) \). (we see \( a_n = e^{b_n} \))

\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{\ln(n)}{n} = \frac{\infty}{\infty}.
\]

Since \( \lim_{t \to \infty} \frac{\ln(t)}{t} = \frac{1}{1} = 1 \),

\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{\ln(n)}{n} = \frac{\infty}{\infty} = 0.
\]

Since \( a_n = e^{b_n} \)

we see \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} e^{b_n} = e^{0} = e^0 = 1 \).

**Monotone Sequences**

**Defn**: \( \{a_n\}_{n \geq 1} \) is monotonically increasing if

\[ a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \]

\( \{a_n\}_{n \geq 1} \) is monotonically decreasing if

\[ a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots \]

and we say \( \{a_n\}_{n \geq 1} \) is monotone if it is monotonically increasing or decreasing.
Then

* If \((a_n)_{n=1}^\infty\) is monotonically increasing and unbounded, then \(\lim_{n \to \infty} a_n = \infty\).

* If \((a_n)_{n=1}^\infty\) is monotonically decreasing and unbounded, then \(\lim_{n \to \infty} a_n = -\infty\).

Then: \(\{a_n\}_{n=1}^\infty\) is bounded AND monotone then \(\{a_n\}_{n=1}^\infty\) converges.

ex: Check convergence of

\[ b_n = \sum_{k=1}^{n} \frac{1}{2^{k+1}}. \]

So we see \(b_n\) is the sum of positive terms.

That is, \(b_n\) is monotonically increasing.

\[ (b_{n+1} = \sum_{k=1}^{n+1} \frac{1}{2^{k+1}} \quad \Rightarrow \quad b_{n+1} - b_n = \frac{1}{2^{n+2}} > 0) \]

We also see \(\frac{1}{2^{k+1}} \leq \frac{1}{2^k}\) for all \(k \geq 1\).

Then \(0 \leq \sum_{k=1}^{n} \frac{1}{2^{k+1}} (= b_n) \leq \sum_{k=1}^{n} \frac{1}{2^k} \leq 1.\) Check (this is geometric sum).
Thus, \((b_n)\) is bounded & monotone \(\Rightarrow\) \(b_n\) is convergent.

**Iterated maps:**

Some sequences are given by a special type of recursive relation in the form

\[ X_{n+1} = f(X_n) \]

where \(f(x)\) is a given function.

- If we have a starting pt \(x_0\)

\[ X_0 \rightarrow X_1 = g(x_0) \rightarrow X_2 = g(g(x_0)) \rightarrow \ldots \rightarrow X_n = g(g(g(\ldots (g(g(x_0))\ldots))) = g^n(x_0). \]

\(\Rightarrow\) \(X_n = \underbrace{g(g(g(\ldots (g(g(x_0))\ldots)))}_{n\text{-many } g's.} \)

**Example:**

\[ X_{n+1} = \frac{1}{2} X_n \quad \text{(we see } X_{n+1} = f(X_n) \text{ where } f(x) = \frac{1}{2}x) \]

\[ \Rightarrow\] \(X_n = \left(\frac{1}{2}\right)^n X_0.\)
We can use geometric means to understand iterated maps.

We use the graphs of $y = g(x)$ & $y = x$, to keep track of $x_0 = v \implies x_1 = g(v), x_2 = g(g(v)), \ldots, x_n = g^n(v)$.

Consider the previous ex $x_{n+1} = \frac{1}{2} x^n$.

$(\Leftarrow) x_0 = v, \; & \; x_{n+1} = g(x_n), \; g(x) = \frac{1}{2} x$

This is called "cobwebbing".

This suggests $\lim_{n \to \infty} x_n = 0$.

\[ \text{EX: } g(x) = \frac{6}{5} x \implies \]

$x_0 = \frac{4}{5}$
$x_{n+1} = g(x_n)$

This suggests $\lim_{n \to \infty} x_n = \infty$. 