Ex: \( \frac{dy}{dt} = y(1-y)(2-y) \)

a) Find all steady states

b) Let \( y_1(t) \) be the soln with \( y_1(0) = \frac{1}{2} \)

Find \( \lim_{t \to \infty} y_1(t) = ? \)

c) Let \( y_2(t) \) be the soln with \( y_2(0) = 3 \)

Find \( \lim_{t \to \infty} y_2(t) = ? \)

L: a) We see the steady states are when \( \frac{dy}{dt} = 0 \) i.e. when \( y = 0, y = 1 \) & \( y = 2 \).

for \( y < 0 \)

Logistic eqn:

\[ N = N(t) \] population of a certain species at time \( t \).

The logistic growth eqn is \( \frac{dN}{dt} = rN_0 \left( \frac{K-N}{K} \right) \) where \( r > 0, K > 0 \) constants.
or \[ \frac{dN}{dt} = r N \left( 1 - \frac{N}{K} \right) = r \left( 1 - \frac{N}{K} \right) N. \]

In comparison, we have the exponential growth which is given by
\[ \frac{dN}{dt} = r N. \quad \text{(so \( N(t) = N(0)e^{rt} \))} \]

Now, let's try to understand the logistic eqn.

\[ \frac{dN}{dt} = r \left( 1 - \frac{N}{K} \right) N. \quad \text{(we have the steady states} \quad N = K \text{&} \quad N = 0). \]

Then

- If \( 0 < N(0) < K \),
  
  Then \( N(t) \) keeps growing (since \( \frac{dN}{dt} > 0 \)).
  
  \[ \begin{align*}
    &\text{But} \quad N(t) < K \\
    \lim_{t \to \infty} N(t) &= K.
  \end{align*} \]

- If \( N(0) > K \), Then \( \frac{dN}{dt} < 0 \) = \( N(t) \) decreasing.
  
  \[ \lim_{t \to \infty} N(t) = K. \]
x \frac{dN}{dt} = 0 \implies N(t) = k \text{ for all } t.
\text{(steady state)}

x \text{ If } N(0) = 0 \text{, then } \frac{dN}{dt} = 0 \implies N(t) = 0.

This value \( k \) is called the carrying capacity and it's related to space, food, etc.

How to solve logistic eqn?

We want to solve

\[
\frac{dN}{dt} = r \left(1 - \frac{N}{k}\right) N
\]

\( N(0) = N_0. \quad \left( \frac{N_0}{N_0 + kc}. \right)

\)

\[
= \frac{d}{dt} \left( \frac{N}{k} \right) = r \left(1 - \frac{N}{k}\right) \frac{N}{k}.
\]

Now, let \( y = \frac{N}{k} \).

we get \( \frac{dy}{dt} = r(1-y)y \) \( \quad y(0) = y_0 = \frac{N_0}{k} \)

To solve \( \frac{dy}{dt} = r(1-y)y \) we use separation of var.

\[
\frac{dy}{(1-y)y} = r \, dt \quad \implies \quad \int \frac{dy}{(1-y)y} = \int r \, dt = rt + c.
\]
How to compute \( \int \frac{dy}{y(1-y)} \)?

We use partial fractions

\[
\frac{1}{y(1-y)} = \frac{A}{1-y} + \frac{B}{y} \implies \frac{1}{y(1-y)} = \frac{Ay + B(1-y)}{y(1-y)}.
\]

\( \implies 1 = (A-B)y + B \) (numerators are same)

for \( y=0 \) we get \( B=1 \).

\( y=1 \) we get \( 1 = A-B+B = A \).

\( \implies \frac{1}{y(1-y)} = \frac{1}{1-y} + \frac{1}{y} \).

\( \implies \int \frac{dy}{y(1-y)} = \int \left( \frac{1}{1-y} + \frac{1}{y} \right) dy = \int \frac{1}{1-y} dy + \int \frac{1}{y} dy.
\]

\( = -\ln|1-y| + \ln|y| = \ln|y| - \ln|1-y|.
\)

\( \implies \) we get \( \ln|y| - \ln|1-y| = c + C \).

\( \implies \ln \left| \frac{y}{1-y} \right| = c + C \implies \left| \frac{y}{1-y} \right| = e^{c+C} = e^c e^C \) constant.

\( \implies \frac{y}{1-y} = e^C e^t \implies \frac{y}{1-y} = B e^t \)
=) If we check the initial cond. we get

\[
\frac{y(0)}{1-y(0)} = Be^0 = B
\]

\[\frac{y}{1-y} = Be^t = y = Be^t(1-y) = y(1+Be^t) = Be^t\]

\[\Rightarrow y = \frac{Be^t}{1+Be^t} \Rightarrow y(t) = \frac{\frac{y_0}{1-\frac{y_0}{1-y_0}} e^t}{1+\frac{y_0}{1-y_0} e^t}\]

\[\Rightarrow y(t) = \left(\frac{1}{1-y_0+y_0e^t}\right) y_0 e^t \quad \text{(recalling \(\frac{N}{\kappa} = y\))}\]

\[\Rightarrow N(t) = \left(\frac{\kappa}{\kappa - N_0 + N_0e^t}\right) N_0 e^t\]