• Cumulative func:

Let \( X \) be a random variable \( a \leq X \leq b \).

Let \( p(x) \) be a probability density func on \( a \leq X \leq b \).

Then we define the cumulative func as

\[
F(t) = \int_{a}^{t} p(s) \, ds \quad (= \mathbb{P}(X \leq t)).
\]

Note:
- \( F'(t) = p(t) \), if \( p(t) \) is cts.
- \( \mathbb{P}(a \leq X \leq b) = \int_{a}^{b} p(s) \, ds \)

FTC:

\[
F(t) = F(b) - F(a)
\]

Example: \( X \) random variable \(-2 \leq X \leq 1\)

\( p(x) \) prob density func. (assume \( p(x) \) is cts)

Which of the followings are not cumulative funcs?

\[ a) \]

No, since it is not decreasing.

\[ b) \]

No, since it shouldn't be negative.
(c) no, since \( F(1) = \int_{-2}^{1} p(s) \, ds \) should be 1.

(d) no, since \( F \) is not cts & not diff.

(e) no, since \( F(-2) \neq 0 \) but \( F(-2) = \int_{-2}^{2} p(s) \, ds \) should be 0.

EX: let \( X \) be a random variable \( 0 \leq x < \infty \).

\[ p(x) = e^{-ax}, \quad a > 0 \text{ constant}, \] is a probability func.

Find \( F(t) = P(X \leq t) \).

\( \) We know \( \int_{0}^{\infty} p(x) \, dx = 1 \) so \( \int_{0}^{\infty} e^{-ax} \, dx = 1 \).

we know \( \int_{0}^{\infty} e^{-ax} \, dx = \lim_{b \to \infty} \int_{0}^{b} e^{-ax} \, dx \)
\[
\int_0^\infty e^{-ax} \, dx = \lim_{b \to \infty} \left[-\frac{1}{a} e^{-ax}\right]_0^{\infty} = \lim_{b \to \infty} \left(\left[-\frac{1}{a} e^{-ab}\right] - \left(-\frac{1}{a}\right)\right)
\]

\[
= \frac{1}{a} = 1 \implies a = 1.
\]

3) \[F(t) = \int_0^t p(s) \, ds = \left[e^{-s}\right]_0^t = e^{-s}\bigg|_0^t = 1 - e^{-t}.
\]

**Mean:** (= Average = Expected Value) & **Median:**

- Random variable \(x\), \(a \leq x \leq b\).
- Prob. density function \(p(x)\).

The **mean** \(\mu\) of the random variable \(x\) is

\[
\mu = \int_a^b x \, p(x) \, dx.
\]

This is the **center of mass** for mass density \(p(x)\).

And the **median** is the value \(x_{\text{med}}\) where \(a \leq x_{\text{med}} \leq b\)

s.t. \[\int_a^{x_{\text{med}}} p(s) \, ds = \frac{1}{2} = \int_{x_{\text{med}}}^b p(s) \, ds = F(x_{\text{med}}).
\]

**Ex:** Bacteria colony:

- Each bacterium can live up to 1000 hrs. (max).
- The prob. for each bacterium to live less than \(t\) hours is \(C \cdot e^{-t + D}\) (\(C, D\) are constants). Find the expected life time & the median life time for a bacterium.