Recall, we haven't calculated the integral

\[ P = \int_0^1 2\sqrt{1+t^2} \, dt. \]

**Differentials:**

\[ \frac{dy}{dx} \approx f'(x). \]

\[ dy \approx f'(x) \cdot dx. \]

Note: This approximation gets better and better as \( \Delta x \to 0 \) and \( \Delta y \to 0 \).

Thus, instead of using the notations \( \Delta x \to 0 \) and \( \Delta y \to 0 \), we will use \( dx, dy \) to denote the infinitesimal change in \( x \) and \( y \):

\[ dy = f'(x) \, dx \quad (\approx \frac{dy}{dx} : f'(x)). \]

These \( dx \) and \( dy \) are called differentials.

**Note:** Differentials \( dx, dy, dz, \ldots \) are not numbers, but their ratios may represent numbers, how much a variable changes with respect to another variable.

\[ \begin{align*}
\text{Ex: } & \quad dy = 3 \, dx \Rightarrow \frac{dy}{dx} = 3, \\
& \quad df = x^2 \, dx \Rightarrow \frac{df}{dx} = x^2
\end{align*} \]
These differentials follow derivative rules:

- If $y = f(x)$, then $dy = f'(x)dx$
- $d(f(x)) = f'(x)dx$
- $C$ constant $\Rightarrow dC = 0$
- $C$ constant $d(Cu) = Cdu$

- $d(u+v) = du + dv$
- $d(uv) = udv + vdu$ (comes from $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$)

- Since $dF(x) = F'(x)dx$, we see:
  \[
  \int_{x=a}^{x=b} dF(x) = \int_{x=a}^{x=b} F'(x)dx = F(b) - F(a).
  \]

- $\int dF = F(x) + C$ - general antiderivative form.

Substitution technique:

Let's go back to our previous ex.

\[
\int_{0}^{2} \sqrt{1+x^2} dx.
\]

(Question: Can we write this integral as $\int_{A}^{B} G(u)\,du$ where $u$ is a new variable s.t. $u = g(x)$ and thus the integral $\int_{A}^{B} G(u)\,du$ is easier?)
let's try $u = 4r^2$, then $du = d(4r^2) = 8r\, dr$.

This way, we see

$$
\int_0^1 2\, r\, dr = \int_0^1 \sqrt{u} \, du
$$

Then $u = 4r^2$ starts from 1

& $u = 4r^2$ goes to 2

$$
\int_0^1 2\, r\, dr = \int_0^2 \sqrt{u} \, du \quad \text{(recall $\frac{du}{dr} = 2r = \frac{du}{2\sqrt{u}}$)}
$$

$$
= \left[ \frac{2}{3} u^{3/2} \right]_1^2 = \frac{2}{3} [2^{3/2} - 1]
$$

Recap: The key point in this example was to see

$$
2r \sqrt{4r^2} \, dr = \sqrt{4r^2} \cdot 2r \, dr
$$

ex: $\int \frac{\sin^2 x \cos x}{u^2} \, du$

We call $\sin x = u \Rightarrow du = d(\sin x) = \cos x \, dx$.

$$
\int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{u^3}{3} + C
$$

$$
= \frac{\sin^3 x}{3} + C. \quad \text{(exercise, check derivative)}
$$
ex: \( \int \frac{x}{1 + x^2} \, dx = ? \)

we see calling \( n = 1 + x^2 \), \( du = d(1 + x^2) = 2x \, dx \).

\[ x \, dx = \frac{du}{2} \]

\[ \int \frac{x}{1 + x^2} \, dx = \int \frac{1}{1 + x^2} \, x \, dx = \int \frac{1}{u} \cdot \frac{du}{2} = \frac{1}{2} \int \frac{1}{u} \, du \]

\[ = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |1 + x^2| + C. \]

ex: \( \int \sin^2 x \, \cos^3 x \, dx = ? \)

\[ = \int \sin^2 x \, \cos^2 x \, \cos x \, dx = \int \sin^2 x \, (1 - \sin^2 x) \, \cos x \, dx \]

\[ = \int (\sin^2 x - \sin^4 x) \, \cos x \, dx \quad \text{call } u = \sin x \]

rest exercise.