we want to show \( \sqrt{xy} \leq \frac{x+y}{2} \) i.e.
\[
2\sqrt{xy} \leq x+y \text{ or in other words}
\]
0 \leq x-2\sqrt{xy}+y. Now it looks something familiar. We see that this expression looks like
\[
x-2\sqrt{xy}+y = (\sqrt{x})^2 - 2\sqrt{x} \sqrt{y} + (\sqrt{y})^2
\]
= (\sqrt{x}-\sqrt{y})^2.

Now the proof is easy.

**Proof:** For \( x, y \) positive, since the square of every number is nonnegative, we get
\[
(\sqrt{x}-\sqrt{y})^2 \geq 0.
\]
This implies \( x-2\sqrt{xy}+y \geq 0 \) i.e. \( x+y > 2\sqrt{xy} \)
and therefore \( \frac{x+y}{2} > \sqrt{xy} \).

\( \square \) (this square empty or full) means the proof is over.

This proof also tells you that \( \frac{x+y}{2} = \sqrt{xy} \) iff \( x=y \) (since \( (\sqrt{x}-\sqrt{y})^2 \geq 0 \) iff \( \sqrt{x} = \sqrt{y} \) i.e \( x=y \)).

**Using Cases:** Sometimes to prove a statement, we can just show that for all possible cases, the statement is true.

**ex:** Proposition: If \( n \in \mathbb{N} \), then \( 4|(-1)^n(2n-1) \) is a multiple of 4.

**Proof:** Suppose \( n \in \mathbb{N} \). Then we know \( n \) is either even or odd. Now consider these two cases.
Case 1: Suppose n is even. Then \( n = 2k \) for some \( k \in \mathbb{Z} \), and \((-1)^k = 1\). Thus, \(1 + (-1)^n (2n-1) = 1 + 1 (2(2k) - 1) = 4k\), which is a multiple of 4.

Case 2: Suppose n is odd. Then \( n = 2k + 1 \) for some \( k \in \mathbb{Z} \), and \((-1)^n = -1\). Thus \(1 + (-1)^n (2n-1) = 1 - 2 (2(k+1) - 1) = -4k\), which is a multiple of 4.

Since n cannot be anything else, we have exhausted all the possible cases for n, and in all these cases (in this case, 2 cases), \(1 + (-1)^n (2n-1)\) is a multiple of 4. Therefore \(1 + (-1)^n (2n-1)\) is a multiple of 4 for every \(n \in \mathbb{N}\).

For this specific example, we can even show:

**Example** Proposition: Every multiple of 4 equals to \(1 + (-1)^n (2n-1)\) for some \(n \in \mathbb{N}\).

Proof: We first need to see this in the "if—then" statement. The proposition says: If \( k \) is a multiple of 4 (i.e., if \( k = 4a \) for some \( a \in \mathbb{Z} \)), then \( k = 1 + (-1)^n (2n-1)\) (or equivalently \(4a = 1 + (-1)^n (2n-1)\) for some \(n \in \mathbb{N}\)).

The cases we can consider are:

- \(a = 0\), \(a > 0\) and \(a < 0\).

Case 1: \(a = 0\) \(\Rightarrow\) \(1 + (-1)^n (2n-1) = 0\) \(\Rightarrow\) \(n = 1\) satisfies this equality.

Case 2: \(a > 0\) \(\Rightarrow\) We can pick \(n = 2a\), then

\[
1 + (-1)^n (2n-1) = 1 + 1 (2(2a) - 1) = 4a = k
\]
Case 3: \( a < 0 \) \( \Rightarrow \) we can pick \( n = 1-2a \). Then we get
\[
1 + (-1)^n (2n-1) = 1 - 1 (2(1-2a) - 1) = 4a - 2.
\]

Therefore, we see that every multiple of 4 can be written as \( 1 + (-1)^n (2n-1) \) for some \( n \in \mathbb{N} \) \( \Box \)

4.5. Treating similar cases: When, in a proof, we have more than 2 cases, but some of these cases are treated similarly, we can omit writing all those cases, and we can say “without loss of generality” and write the proof of only one of those similar cases. But, we need to be careful! Those cases we claim to be similar may not be similar. So, before writing “Without loss of generality” or shortly “WLOG”, we need to make sure they are indeed treated similarly. Also, if we have more than two cases, some of which are treated similarly, we need to make sure which ones are similar.

**Example Proposition:** \( \forall x, y \in \mathbb{R} \), \( |x+y| \leq |x| + |y| \) (called the triangle’s inequality. This inequality is very important and whenever we define a new “distance” (A.K.A. a metric) we need it to satisfy triangle’s inequality.)

**Proof:** To prove this, we need to look at different cases.

---

Case 1: Both \( x \) and \( y \) are nonnegative. Then we know \( |x| = x \) and \( |y| = y \). Moreover we know \( xy \geq 0 \) and thus \( |x+y| = xy = |x| + |y| \).
Case 2: Both $x$ and $y$ are negative. Then
\[ |x| = -x \text{ and } |y| = -y. \]
Moreover $x < 0$ and $y < 0$ and thus
\[ |x + y| = -(x + y) = (x) + (-y) = |x| + |y|. \]
which is also what we wanted to show.

Case 3: Only one of $x$ or $y$ is negative.
Since the case is symmetric in $x$ and $y$, we see that WLOG we can assume $x < 0$ and $y > 0$.
Now for this case, we need two sub cases.

Case 3a: $|x| > |y|$ (e.g. $x = -5$, $y = 3$)
Then we see $x + y < 0$. This implies
\[ |x + y| = -(x + y) = (-x) - y = |x| - |y|. \]
Since we know $|y| > 0$, we see $|x| - |y| \leq |x| - |y| + 2|y| = |x| + |y|$. Therefore
\[ |x + y| \leq |x| + |y|. \]

Case 3b: $|x| \leq |y|$.
In this case we see $x + y > 0$ and hence
\[ |x + y| = x + y = y - (-x) = |y| - |x| \leq |y| + |x| \] (Similar to the previous case).
Therefore, for all possible cases $|x + y| \leq |x| + |y|$.

Chapter 5: Contrapositive proof.

If we have a proposition of the form $P \Rightarrow Q$. Sometimes trying to prove this directly can be a