Chapter 2: Logic

In mathematics, we have axioms and using the axioms we have, we prove "facts" such as lemmas, theorems etc. To be able to do that we need a way to "combine" the facts we have, while keeping their truth value. This "way" is called logic. Using logic, we will look at how sentences "combine" to make another sentence and we will also be able to understand how this new sentence is related to its individual pieces.

2.1: Statements.

**Defn:** A statement is a sentence or a mathematical expression that is either definitely true or definitely false. Thus, statements are going to be the "sentences" we talked above that we are going to apply logic to produce other information.

**Ex:** If a circle has radius $r$, then its area is $\pi r^2$ square units. (true statement)

- $2 \in \mathbb{Z}$ (reads as: 2 is an element of the set of integers) (true statement)
• $5=2$ (reads as: 5 is equal to 2) (false statement)

• Some right triangles are equilateral (false statement)

• Add 5 to both sides (not a statement, since we cannot talk about whether or not this sentence is true or not. But, if we say "adding 5 to both sides of $x-5=37$ gives $x=42" is a statement.

We said that a sentence is a statement if it is true or false. This means that if a sentence has to be either true or false, it still is a statement even when we don't know whether or not it is true or false. This may sound a bit confusing, but if we consider the example, it will become more clear:

"Every even integer greater than 2 can be written as the sum of two primes."

This sentence has to be true or false, although, as of this moment, no one knows whether it is true or not, so this sentence is a
statement (in fact, it is quite a famous statement; and is called "Goldbach's conjecture". In fact, a conjecture is a statement claimed to be true, although we don't know whether it is or not.)

In logic, when we work with statements, it is often practical to use letters for specific statements. This way we can avoid writing the same sentence over and over again.

**ex:** P: For every integer \( n > 1 \), the number \( 2^{n-1} \) is prime. (False statement)

Q: Every polynomial of degree \( n \) has at most \( n \) roots (true statement).

Here, as we did with the sets, we can use indices to denote different statements.

**ex:** \( S_1: \mathbb{Z} \subseteq \emptyset \) (reads as: set of integers is a subset of empty set) (false statement) and

\( S_2: \{0, -1, -2\} \cap \mathbb{N} = \emptyset \) (reads as: the intersection of the set \( \{0, -1, -2\} \) and \( \mathbb{N} \) is empty) (true statement)
The statements can contain variables, but not every sentence that contains variables are statements.

ex: P: "If an integer \( x \) is a multiple of 6, then \( x \) is even" is a true statement whereas

Q: The integer \( x \) is even" is not a statement since the "truth value" of this sentence depends on what "\( x \)" is.

A sentence, whose truth ("truth value") depends on the value of one or more variables, is called an "open sentence".

2.2 And, or, not:

We can use the words "and" and "or" to combine two statements to form a new statement.

ex: P: The number 8 is even and a power of 2. can be seen as the combination of the two statements

R: The number 8 is even
Q: 8 is a power of 2. with the word "and". Since we are using letters to denote sentences, we are also going to introduce new