A combination, on a combination lock, say "turn right to 3, then left to 1, then right to 5" can be seen as an ordered triplet $(R, 3, (L, 1), (R, 5)) \in (D \times V)^3$.

**Exercise:** Try to illustrate, in $\mathbb{I}^2$, the sets $\mathbb{N} \times \mathbb{Z} \times \mathbb{I}$ and $\mathbb{Z} \times \mathbb{I} \times \mathbb{Z}$.

1.3 Subsets.

We saw what it means for an object to be an element of a set. In this chapter, we will see what it means for a set to be contained in another set.

**Defn:** A set $A$ is called a subset of a set $B$ if every element of $A$ is an element of $B$. We denote it as $A \subseteq B$ or $B \supseteq A$.

If $A \subseteq B$ and $B$ has an element that is not in $A$ then we say $A$ is a proper subset of $B$ and we denote it as $A \subset B$ or $B \supset A$.

Similarly, if there is an element of $A$ that is not in $B$, we say $A$ is not a subset of $B$ & denote it as $A \not\subseteq B$ or $B \not\supset A$.

**Note:**
- $\emptyset \subseteq A$ for any set $A$.
- If $A \subseteq B$ and $B \subseteq A$ then $B = A$. 

ex: Find all subsets of \( S = \{0, 1, 2\} \)

\( \emptyset \) (since emptyset is a subset of every set)

\( \{0\}, \{1\}, \{2\}, \{0, 1, 2\} \) so there are four subsets of \( S = \{0, 1, 2\} \).

ex: \( \{2, 3, 7\} \neq \{2, 4, 5, 6, 7\} \)

- \( \{2n : n \in \mathbb{Z}\} \leq \mathbb{Z} \)
- \( \mathbb{R} \times \mathbb{N} \leq \mathbb{R} \times \mathbb{Z} \leq \mathbb{R} \times \mathbb{R} \).

ex: Find all subsets of \( \bar{S} = \{0, 1, 2\} \).

We see that \( S = \{0, 1, 2\} \subset \bar{S} \) and thus every subset of \( S \) is a subset of \( \bar{S} \) i.e. \( \emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\} \) which are the subsets that don't have the element "2" in them. So if we consider all the subsets that have "2" in them we get

\( \{2, 3\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\} \) (4 such subsets i.e. same number of subsets of \( S \))

This implies \( \bar{S} \) has 8 subsets which is \( 2^3 \) (number of subsets of \( S \))

From this we can guess that:

- A set that has \( n \) elements has \( 2^n \) subsets.

(We will be able to prove this easily when we learn how to "count".)
**Defn:** The set of all subsets of a set is called a power set of \( A \), and denoted by \( P(A) \).

**Ex:** Let \( S = \{1\} \), then \( P(S) = \{ \emptyset, \{1\} \} \) and \( P(P(S)) = \{ \emptyset, \emptyset, \emptyset, \emptyset, \{1\}, \{\emptyset\}, \{1, \emptyset\} \} \).

Here we need to be careful with sets & the elements since the elements of \( P(P(S)) \) are sets of sets (and hence, the double braces)

**Ex:** \( P(\{1, 2, 3\}) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \} \)

**Note:** • \( \{1, 2\} \not\in P(\{1, 2, 3\}) \) since \( \{1, 2\} \not\in \{1, 2, 3\} \), since \( 2 \not\in \{1, 2, 3\} \)

• \( P(S) \) is defined for Sets \( S \), thus, \( P(\{1\}) \) does not mean anything since 1 is an element but not a set.

1.5 Unions, Intersections & Difference

In the previous section, we discussed the sets & subsets. This, however, is not enough when we want to talk about two sets \( A \& B \) in terms of their elements, such as, "what are the elements that \( A \) has but \( B \) does not?" or "what are the elements they share?", etc.

In this section we are going to define operations that solve this problem.
Define: Suppose $A$ & $B$ are sets

The union of $A$ & $B$ is the set $A \cup B = \{ x: x \in A \text{ or } x \in B \}$.

The intersection of $A$ & $B$ is the set $A \cap B = \{ x: x \in A \text{ and } x \in B \}$.

The difference of $A$ and $B$ is the set $A - B = \{ x: x \in A \text{ and } x \not\in B \}$.

Ex: Suppose $A = \{a, b, c, d, e, f\}$, $B = \{d, e, f\}$, $C = \{1, 2, 3\}$

Then $A \cup B = \{a, b, c, d, e, f\}$
$A \cap B = \{d, e, f\}$
$B - A = \{f\}$.

What about $(A \cap C) \cup (A - B)$?

or $(A \times C) \cap (B \times C)$?

or $(A \times C) \cap (C \times B)$?

Ex: $A = \{(x, x^2): x \in \mathbb{R}\}$ (the graph of the function $y = x^2$)

and $B = \{(x, x+2): x \in \mathbb{R}\}$ (the graph of $y = x+2$)

Then $A \cap B = \{(1, 3), (2, 6)\}$
$A - B = \{(x, x^2): x \in \mathbb{R} \text{ and } x \not\in \mathbb{R} - \{1, 2\}\}$

where $A \cap B$ is the set of points that are shared by both graphs whereas $A - B$ is the graph $A$ except for points shared with $B$. 

We also have a name for the set of all elements that are not in a set \( A \). But for this we need to specify where the set \( A \) lies in. For example, if we want to write down the set, \( A \), the set of all people who are not wearing anything blue today, this set is the set of all people who are not in the set, \( B \), the set of all people who wear something blue today. For the answer we need to say whether we talk about people in this room or people in this campus or any group of people actually. This set of "people in this room" or "people in campus" is called a universal set.

**Defn:** Given a universal set \( U \) and a set \( A \subset U \), we define the complement of \( A \) denoted by \( \overline{A} \) as the set of all elements not in \( A \).

\[
\overline{A} = \{ x \in U : x \notin A \}
\]

since this is our universal set

This defn allows us to write the expressions like \( A-B \) in terms of intersections and unions.

\[ e.g. \ A - B = A \cap \overline{B} \]

ex: If \( P \) is the set of primes (as a subset of \( \mathbb{N} \))
then \( \overline{P} = \mathbb{N} - P = \{1, 4, 6, 8, 9, 10, \ldots \} \) which is the set of all...
composite numbers and 1.

One of the easier ways to visualize all these set operations is to draw what is called a Venn diagram where you draw a set as a circle.

The set of elements of B that is not in A
\[\text{i.e. } B - A\]

The set of elements of A that are not in B
\[\text{i.e. } A - B\]

The set of elements shared by A and B, i.e. \[A \cap B\].

And \[A \cup B\] is the set of all elements that are in either A or B or both, i.e. ones in blue shaded region, red shaded region or black shaded region. Thus, all shaded regions combined is \[A \cup B\].

You can do this with as many sets as possible (although it may get a little confusing after 4 sets).

\[\text{ex:}\]

\[A \cup B \cup C\] or \[(B \cap C) - A\] or \[(B \cap C) - (A \cap B \cap C)\]
ex: Make a Venn diagram for 

\((A \cup (B \cap C)) - (A \cap B \cap C)\) Now I admit this looks a little complicated, but all we need to do is to operations and draw the diagram step by step.

1. \(A \cup (B \cap C)\)

and recall that union means shaded altogether.

this means \(A \cup (B \cap C)\) is

recall that sets don't care about how many times an element is written, hence the red and blue shaded region above counts as one.

Now shade above \(A \cap B \cap C\) i.e. the set of elements in \(A, B, C\)

Now recall that difference means we don't take any elements from the second set in the difference. i.e. we are to take no element from the red region.
which means

\[(A \cup (B \cap C)) - (A \cap B \cap C)\]  

\[\text{c} \quad c\]
\[\text{A} \quad \text{B}\]

**Remarks:** The same set may have different looking expressions.

ex: \[A \quad B\]  
The shaded set is \(A-B\) and also \(A - (A \cap B)\). So, just because they look different does not mean that they are different.

If we have a universal set, the sets given, i.e. in we draw it as a set that encloses those sets.

ex: \[\text{ex:}\]

\[\text{ex:}\]

shaded region \[\overline{(A \cap B)}\]