Some remarks on Probability

Let \( f(x) = \begin{cases} \frac{1}{20}, & 0 \leq x \leq 20 \\ 0, & \text{otherwise} \end{cases} \)

Then

\[ P(x = 7) = 0 \quad \text{even if} \quad f(7) = \frac{1}{20} \]

Indeed, there are infinitely many numbers between 0 and 20. So the chance of choosing any specific number \( x \) \((0 \leq x \leq 20)\) is 0:

\[ P(x = a) = 0 \]

But the chance of choosing a number between 5 and 8 is \( \frac{3}{20} \):

\[ P(5 \leq x \leq 8) = \int_{5}^{8} f(x) \, dx = \int_{5}^{8} \frac{1}{20} \, dx = \left[ \frac{1}{20} x \right]_{5}^{8} = \frac{8}{20} - \frac{5}{20} = \frac{3}{20} \]

Properties of a P.D.F. \( f(x) \):

(i) \( f(x) \geq 0 \) for all \( x \)

(ii) \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \)

The Cumulative distribution function:

\[ F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t) \, dt \]

(The area function of \( g(t) = f(t) \))

By F.T.C. \( F(x) = F(x) \)

\[ P(c \leq X \leq d) = \int_{c}^{d} f(x) \, dx = \int_{c}^{d} f(x) \, dx - \int_{-\infty}^{c} f(x) \, dx = F(d) - F(c) \]

The expected value (mean) is an analogue for the average:

\[ E(x) = \int_{a}^{b} x \, f(x) \, dx \]
For example, consider this collection of numbers:

\[ 2, 2, 3, 0, 2, 3 \]

Their average is \( \frac{2 + 2 + 3 + 0 + 2 + 3}{6} = 2 \)

One can order the numbers and interpret the average as follows:

\[
\text{average} = \frac{0 + 2 + 2 + 2 + 3 + 3}{6} = \frac{1}{6}(0) + \frac{3}{6}(2) + \frac{2}{6}(3)
\]

the chance of choosing 0 among these 6 numbers

the chance of choosing 2

the chance of choosing 3.

\[ = P(X=0) \cdot (0) + P(X=2) \cdot (2) + P(X=3) \cdot (3) \]

The above was an example for a discrete random variable \( X \). For a continuous random variable, the average looks similar:

\[ E(x) = \int_{a}^{b} x f(x) \, dx \]

\[ \text{the chance of choosing } x \]

* \( \text{Var}(X) = \int_{a}^{b} (x - \mu)^2 f(x) \, dx \) \quad \text{"Variance"} \]

\[ \left( \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx \right) \quad \text{if } f \text{ is defined only for } a \leq x \leq b \]

\[ \left( \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx \right) \quad \text{if } f \text{ is defined for all } x \]

where \( \mu = E(x) \).

So, in order to compute \( \text{Var}(X) \), we need to compute first \( E(x) \).

* A second formula for \( \text{Var}(X) \):

\[ \text{Var}(X) = \int_{a}^{b} x^2 f(x) \, dx - \mu^2 \]

* The standard deviation \( \sigma(X) = \sqrt{\text{Var}(X)} \)