§ 9.3 Taylor Series

Def. Suppose $f$ is a "nice" function (which means that $f$ has derivatives of all orders) on an interval centered at $x = a$:

Taylor series for $f$ centered at $a$ is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

A Taylor series centered at 0 (i.e., $a=0$) is called a Maclaurin series.

Ex. Find the Maclaurin series for the following functions and find their interval of convergence.

a) $f(x) = \cos x$

$f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

$f^{(4)}(x) = \cos x \Rightarrow f^{(4)}(0) = 1$  (derivatives are periodic: $f^{(5)}(x) = f^{(1)}(x) = \cdots$)

The Maclaurin series of $\cos x$ is

$$\sum_{k=0}^{\infty} c_k x^k$$

where $c_k = \frac{f^{(k)}(0)}{k!}$  

We see that $c_k = \frac{f^{(k)}(0)}{k!} = 0$ when $k$ is odd.

We have

$c_0 = f(0) = 1$, $c_4 = \frac{f^{(4)}(0)}{4!} = \frac{1}{4!}$, $\ldots$

$c_2 = f''(0) = -1$, $c_6 = \frac{f^{(6)}(0)}{6!} = \frac{-1}{6!}$, $\ldots$

So

$$\sum_{k=0}^{\infty} c_k x^k = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

For what value of $x$ does the series converge? We apply the Ratio test:

$$r = \lim_{k \to \infty} \left| \frac{(-1)^{k+1} 2^{k+1}/(2k+1)!}{(-1)^k 2^k/(2k)!} \right| = \lim_{k \to \infty} \frac{2^{k+1}}{(2k+1)!} \left( \frac{2k}{2k+1} \right) = \lim_{k \to \infty} \frac{2}{2k+1} = 0 < 1$$

(\text{because $|x^2|$ is fixed since $x$ is fixed}) So the Maclaurin series converge absolutely for all $x$. (Because $r = 0 < 1$ no matter what $x$ is.) We conclude that the interval of convergence is $-\infty < x < \infty$.

Remark: $(2k+1)! = (2k+2)! = 1 \times 2 \times \cdots \times (2k) \times (2k+1)$}

$$= (2k)! \cdot (2k+1)(2k+2)$$
b) \( f(x) = \frac{1}{1-x} \)

\[
\begin{align*}
\frac{f(x)}{1-x} & \Rightarrow f(0) = 1 \\
\frac{f'(x)}{(1-x)^2} & \Rightarrow f'(0) = 1 = 1! \\
\frac{f''(x)}{(1-x)^3} & \Rightarrow f''(0) = 2 = 2! \\
\frac{f'''(x)}{(1-x)^4} & \Rightarrow f'''(0) = 3 \times 2 = 3! \\
\frac{f^{(4)}(x)}{(1-x)^5} & \Rightarrow f^{(4)}(0) = 4 \times 3 \times 2 = 4!
\end{align*}
\]

and in general \( f^{(k)}(0) = k! \),

\[
C_k = \frac{f^{(k)}(0)}{k!} = \frac{k!}{k!} = 1 \quad \text{for} \quad k = 0, 1, 2, \ldots
\]

"The Maclaurin series of \( f(x) = \sum_{k=0}^{\infty} C_k x^k = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots \)"

This geometric series converges for \( x \) such that \( |x| < 1 \).

\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots
\]

So \( f(0) = 1 \) which we already know, since \( f(0) = \frac{1}{1-0} \)

\[
\begin{align*}
f(0) & = 1 \quad \text{true} \\
f'(0) & = 1! \times 1 = 1 \\
f''(0) & = 2! \times 1 \\
f'''(0) & = 3! \times 1 = 6 \\
f^{(4)}(0) & = 4! \times 1 = 24
\end{align*}
\]

We got all the derivatives of \( f \) at \( x = 0 \) \( \{ f^{(k)}(0) \} \) for free, by looking at its Maclaurin series \( f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \)
Example for Manipulating Maclaurin Series: Let \( f(x) = e^x \)

a) Find the Maclaurin Series for \( f \):

Maclaurin Series for \( f = \sum_{k=0}^{\infty} c_k x^k \)

(\[ c_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!} \])

\[ = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + x^2/2! + x^3/6 + \ldots \]

b) Find the interval of convergence:

\[ r = \lim_{k \to \infty} \left| \frac{x^{k+1}/(k+1)!}{x^k/k!} \right| \]

\[ = \lim_{k \to \infty} \left| \frac{x}{(k+1)!/k!} \right| = \lim_{k \to \infty} \left| \frac{x}{k+1} \right| \]

(\( x \) is fixed)

\[ = 0 < 1 \]

\( \Rightarrow \) the series \( \sum \left| \frac{1}{k!} x^k \right| \) no matter what \( x \) is

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\( \Rightarrow \) the interval of convergence of \( \sum \frac{1}{k!} x^k \) is \( (-\infty, \infty) \)

c) "The Maclaurin for \( a^x e^x \) = \( e^x \sum_{k=0}^{\infty} \frac{a^k}{k!} x^k \)

\[ = e^x \left( 1 + x + x^2/2 + x^3/6 + \ldots \right) \]

\[ = a^x + a^x x^2/2! + a^x x^3/6 + \ldots \]

\[ = \sum_{k=0}^{\infty} \frac{a^k}{k!} x^k \]

\[ \text{have the interval of conv of } (-\infty, \infty) \]

"The Maclaurin series for \( e^{-x^2} \) = \[ \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^k}{k!} \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} \]

\[ = 1 - x^2 + x^4/2! - x^6/3! + \ldots \]

It converges if \( x^2 \in (-\infty, \infty) \), so \( x \in (-\infty, \infty) \) (\( x \) can be anything)

(\( \text{the int of conv of } e^{x^2} \))

\( \Rightarrow \) the int of conv of the Maclaurin series for \( e^{-x^2} \) is \( (-\infty, \infty) \).
Remark: Not only Taylor series of a function \( f(x) \) are approximations for \( f(x) \), the full Taylor series (with all its infinitely many terms) is equal to \( f(x) \) for most of functions we work with. For example

\[
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots
\]

For more see table 9.5 on page 694.

\[\text{§ 9.4 Working with Taylor Series}\]

We can solve many problems easily by using Taylor series.

Ex: Compute the following limits:

a) \[
\lim_{x \to 0} \frac{x^2 + 2 \cos x - 2}{3x^4}
\]

\[
= \lim_{x \to 0} \frac{x^2 + 2(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots) - 2}{3x^4}
\]

\[
= \lim_{x \to 0} \frac{x^2 + \frac{x^4}{12} - \frac{x^6}{360} + \ldots}{3x^4}
\]

\[
= \lim_{x \to 0} \left( \frac{1}{36} - \frac{x^2}{1080} + \ldots \right) = \frac{1}{36}
\]

b) \[
\lim_{x \to 0} \left( 6x^5 \sin \frac{1}{x} - 6x^4 + x^2 \right) = l
\]

Do the substitution \( t = \frac{1}{x} \) so that \( t \to 0 \) when \( x \to \infty \), and then \( \sin \frac{1}{x} = \sin t \) and we can use the Taylor series (indeed MacLaurin series) of \( \sin t = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \ldots \)

\[
\Rightarrow l = \lim_{t \to 0} \left( 6 \left( \frac{1}{t} \right)^5 \sin t - 6 \left( \frac{1}{t} \right)^4 + \left( \frac{1}{t} \right)^2 \right) = \lim_{t \to 0} 6 \sin t - 6t + t^3
\]

\[
= \lim_{t \to 0} \frac{6 \left( t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \ldots \right) - 6t + t^3}{t^5}
\]

\[
= \lim_{t \to 0} \left( \frac{1}{20} - \frac{t^2}{840} + \ldots \right) = \frac{1}{20}
\]