\[ \begin{align*}
& f_{xx} = \frac{-y \cdot 2x (1 + x^2 y^2)}{(1 + x^2 y^2)^2} = \frac{-y (2xy^2)}{(1 + x^2 y^2)^2} \\
& f_{yy} = \frac{-x (2x^2 y)}{(1 + x^2 y^2)^2} \\
& f_{xy} = \frac{1}{1 + x^2 y^2} - \frac{y (2x y)}{(1 + x^2 y^2)^2}
\end{align*} \]

\[ \begin{align*}
& f_{xx}(0,0) = 0 \quad \Rightarrow D(0,0) = 0 \cdot 0 - 1^2 = -1 < 0 \quad (0,0) \text{ saddle}
\end{align*} \]

c) \[ f(x, y) = \sqrt{x^2 + y^2 - 4x + 5} \]
\[ \begin{align*}
& f_x = \frac{2x - 4}{\sqrt{x^2 + y^2 - 4x + 5}} = \frac{x - 2}{\sqrt{x^2 + y^2 - 4x + 5}} \\
& f_y = \frac{2y}{\sqrt{x^2 + y^2 - 4x + 5}} \\
\end{align*} \]

\[ \begin{align*}
& f_{xx} = \frac{1}{\sqrt{x^2 + y^2 - 4x + 5}} + \frac{(2x - 4)(1)}{2(x^2 + y^2 - 4x + 5)^{3/2}} \\
& f_{yy} = \frac{2}{\sqrt{x^2 + y^2 - 4x + 5}} + \frac{-y (2x - 4)}{(x^2 + y^2 - 4x + 5)^{3/2}} \\
& f_{xy} = \frac{(x - 2)(-1/2)(2y)}{(x^2 + y^2 - 4x + 5)^{3/2}}
\end{align*} \]

\[ \begin{align*}
& f_{xx}(2,0) = 2 + 0 = 2 \\
& f_{yy}(2,0) = 1 \\
& f_{xy}(2,0) = 0 \\
& \Rightarrow D(2,0) = 2 \cdot 1 - 0^2 = 2 > 0 \quad f_{xx}(2,0) = 2 > 0 \quad \text{local minimum}
\end{align*} \]

Def: Absolute maximum value: The largest value for \( f \)

Absolute minimum value: The smallest value for \( f \)
Procedure: Finding absolute Max./Min. values on closed boundary set.

Let \( f \) be continuous on a closed set \( R \subset \mathbb{R}^2 \):
1) Determine the value of \( f \) at all critical points in \( R \).
2) Find the max. & min. values on the boundary of \( R \), important.
3) Compare what you get in 1) and 2) to find the absolute Max./Min.

Ex (Page 944): \( f(x,y) = xy - 8x - y^2 + 12y + 160 \) over the triangular region \( R = \{(x,y) : 0 \leq x \leq 15, 0 \leq y \leq 15-x\} \)

1) \( f_x = y - 8 \)
   \( f_y = x - 2y + 12 \)
   \( \begin{cases} f_x = y - 8 = 0 \\ f_y = x - 2y + 12 = 0 \end{cases} \Rightarrow \begin{cases} y = 8 \\ x = 2y - 12 = 4 \end{cases} \)
   \( f(4,8) = 192 \)

2) Search for extrema on the boundary of \( R \)
   i) Let \( C_1 = \{(x,y) : y = 0 \text{ for } 0 \leq x \leq 15\} \)
      We define \( g_1(x) = f(x,0) = 160 - 8x \)
      The candidates for extrema of \( g_1 \) are critical points on \( C_1 \)
      and endpoints (\( x = 0 \) and 15)
      \( g_1'(x) = -8 \neq 0 \text{ no critical point} \)
      \( g_1(0) = 160 \text{ max of } f \text{ on } C_1 \)
      \( g_1(15) = 40 \text{ min of } f \text{ on } C_1 \)
   ii) Let \( C_2 = \{(x,y) : x = 0, 0 \leq y \leq 15\} \)
      Define \( g_2(y) = f(0,y) = -y^2 + 12y + 160 \)
      \( g_2'(y) = -2y + 12 = 0 \Rightarrow y = 6 \)
      \( g_2(6) = 352 \text{ max of } f \text{ on } C_2 \)
      \( g_2(0) = 160 \text{ min of } f \text{ on } C_2 \)
\[
g_2'(y) = -2y + 12 = 0 \Rightarrow y = 6 \quad \text{critical point}
\]
\[
\begin{align*}
g_2(6) &= 196 \\
g_2(0) &= 160 \\
g_2(15) &= 115
\end{align*}
\]

iii) \( C_3 = \{(x, y) \mid y = 15 - x, \ 0 \leq x \leq 15\} \)

\[
g_3(x) = f(x, 15 - x) = -2x^2 + 25x + 115
\]

\[
g_3'(x) = -4x + 25 = 0 \Rightarrow x = \frac{25}{4}
\]

\[
\begin{align*}
g_3(0) &= f(0, 15) = 115 \\
g_3(15) &= f(15, 0) = 40 \\
g_3(6.25) &= f(6.25, 8.75) = 193.125
\end{align*}
\]

\[
\Rightarrow \begin{cases}
\text{Abs. min value of } f = 40 \text{ happening at } (15, 0) \\
\text{Abs. max. value of } f = 196 \text{ happening at } (0, 6)
\end{cases}
\]

Ex (P946): \( f(x, y) = x^2 + y^2 - 2x + 2y + 5 \) on the region \( R = \{(x, y) \mid x^2 + y^2 \leq 4\} \)

\[
f(x, y) = \begin{cases} 
  x^2 - 2x + y^2 + 2y + 5 = (x-1)^2 - 1 + (y+1)^2 - 1 + 5 
\end{cases}
\]

1) \( \begin{align*}
  f_x &= 2x - 2 = 0 \\
  f_y &= 2y + 2 = 0 
\end{align*} \Rightarrow \begin{cases} x = 1 \\
  y = -1 
\end{cases} \)

\((1, -1) \) is in \( R \), so we need to consider it: \( f(1, -1) = 3 \)

2) Boundary of \( R \): the circle \( x^2 + y^2 = 4 \).

We can not write \( y \) in terms of \( x \) on \( x^2 + y^2 = 4 \):

\[ y^2 = \sqrt{4-x^2} \]

For each \( x \), there are two \( y \)'s.

One way: Devide the boundary

On \( (i) \):

\[
g_1(x) = (x-1)^2 + (\sqrt{4-x^2} + 1)^2 + 3
\]

\[= -2x + 2\sqrt{4-x^2} + 9 \] and continue.
Better way:
Describe the boundary by
\[
\begin{align*}
x &= 2 \cos \theta, \quad 0 \leq \theta \leq 2\pi \\
y &= 2 \sin \theta
\end{align*}
\]
Define \( g(\theta) = f(2\cos \theta, 2\sin \theta) = \)
\[
\frac{4 \cos^2 \theta + 4 \sin^2 \theta}{4} - 4 \cos \theta + 4 \sin \theta + 5
\]
\[
= -4 \cos \theta + 4 \sin \theta + 9
\]
g'(\theta) = 4 \sin \theta + 4 \cos \theta = 0 \Rightarrow \sin \theta = -\cos \theta
\Rightarrow \tan \theta = -1 \Rightarrow \theta_1 = \frac{3\pi}{4} \Rightarrow g(\theta_1) = 9 - 4\sqrt{2} \approx 3.3
\]
\[
\theta_2 = \frac{7\pi}{4} \Rightarrow g(\theta_2) = 9 + 4\sqrt{2} \approx 14.7
\]
Boundary point is \( \theta \in [0, 2\pi] \): 0 or \( 2\pi \) associated to (2,0)
\[
g(0) = f(2, 0) = 5
\]
Abs. max value \( = 9 + 4\sqrt{2} \) at \((-\sqrt{2}, \sqrt{2})
Abs. min value \( = 3 \) at \((1, -1)

More motivational discussion for derivative

Derivative \( f'(x) \) is a function that helps to understand the shape of \( f(x) \)

Question: What is \( \sin(1^\circ) \)?
\[
\sin(1^\circ) = \sin(\pi/180)
\]
\( \pi/180 \) is close to 0 \( \Rightarrow f(0) = \sin 0 = 0 \)
\( \Rightarrow f(0) = \cos 0 = 1 \)
Near 0, \( f(x) \) is almost equal to \( x \)
\[
f(x) \approx x \quad \text{near} \ x = 0
\]
\[
\Rightarrow \frac{f(\pi/180)}{\pi/180} \approx 1 = 0.01745329...
\]
\[
f'(\pi/180) = 0.01745240...
\]
The same thing for a two-variable function

\[ f(2,1) = 5 \]

How much is \( f(2.1, 1) \)?

\[ f(2.1, 1.2) = ? \]

Assume \( f(2,1) = 3 \)

near (2,1)

\[ f_x(2,1) = -2 \]

\[ \Rightarrow f(x,1) \approx f(2,1) + f_x(2,1) \cdot (x-2) \]

so \( f(2.1, 1) \approx 5 + 3(2.1-2) \)

near (2,1)

\[ = 5, 3 \]

Similarly

\[ f(2, y) \approx f(2,1) + f_y(2,1) \cdot (y-1) \]

so \( f(2,1.2) \approx 5 + (-2)(1.2-1) = 4.6 \)

\[ f(x, y) \approx f(2,1) + 3(x-2) - 2(y-1) \]

\[ f(2.1, 1.2) \approx 5 + 0.3 - 0.4 = 4.9 \]

\[ f(x, y) \approx f(2,1) + \nabla f(2,1) \cdot (x-2, y-1) \]

\[ \nabla f(x, y) = (f_x(x, y), f_y(x, y)) \]

\[ \nabla f(2,1) \cdot (x-2, y-1) = 3(x-2) + (-2)(y-1) \]