Introduction

In the classical setting, algebraic geometry tries to classify (projective) algebraic varieties, i.e. subsets \( X \subset \mathbb{P}^n(K) \) described by algebraic equations, where \( K \) is an algebraically closed field. A crucial objective is, given a specific kind of algebraic varieties, to parametrize their isomorphism classes in some geometrically meaningful way. This is often referred to as a *moduli problem*. Possibly the most famous moduli problem is to parametrize families of smooth curves, which led to the construction of the moduli space \( M_g \) of smooth curves.

A powerful breakthrough in moduli theory was the introduction by Deligne and Mumford [DM69] of the *moduli stack* \( \mathcal{M}_g \) of smooth algebraic curves of genus \( g \), which allowed for a relatively easy proof of their famous result that the variety \( M_g \) is irreducible. It soon became evident that the stack \( \mathcal{M}_g \) was actually a much better object to work with than \( M_g \), giving birth to the widespread use of algebraic stacks.

An algebraic stack is a generalization of a variety, where the points can have “intrinsic automorphisms”. Using them instead of ordinary varieties to represent moduli problems allows to describe the problem much more faithfully, and to have good properties such as smoothness if the objects being described are well-behaved. For example, the moduli stack \( \mathcal{M}_g \) is smooth, while the moduli space \( M_g \) is not. Algebraic stacks produce an algebro-geometric version of orbifolds, and also of classifying spaces \( B\Gamma \) in topology.

The price to pay for these advantages is an increased level of technicality, which requires mastery of both geometric and categorical arguments. Many areas have benefited from the theory of algebraic stacks, from the study of abelian varieties to surfaces, and recently they are becoming of interest to physicists too, for example in string theory.

My work has concentrated on constructing and computing invariants for moduli stacks. Functorial invariants for moduli stacks are especially useful, as they will provide invariants for the families of objects being parametrized. Prime examples of invariants which are of interest are the Picard group of line bundles, the Chow groups (which are an algebraic version of singular homology), and étale cohomology with various kind of coefficients.

Past research

A majority of my research [Pir15, Pir17, Pir16] regards *Cohomological invariants*, a theory of arithmetic invariants which were classically associated with principal \( G \)-bundles over algebraic groups and thus can be regarded as invariants of the moduli stack \( B\Gamma \), which like its topological counterpart classifies principal \( G \)-bundles. I extended the theory to arbitrary algebraic stacks, and then computed the cohomological invariants of the stacks of elliptic curves \( \mathcal{M}_{1,1} \), of the stack of smooth genus two curves \( \mathcal{M}_2 \) and of the stacks \( \mathcal{H}_g \) of smooth hyperelliptic curves of genus \( g \) for all even \( g \) and for \( g = 3 \).

I also studied the Picard groups of universal families of Abelian varieties [FP16] (joint with R.Fringuelli) and the motivic classes of the classifying spaces \( B\Spin_n \) of \( \Spin_n \)-principal bundles [PT17] (joint with M.Talpo).

Present and future research

*Computing the cohomological invariants of \( \mathcal{M}_g \):*

A cohomological invariant \( \sigma \) of a moduli stack \( \mathcal{M} \) can be thought of as an arithmetic equivalent to a characteristic class; given a family of objects \( X \rightarrow S \) parametrized by \( \mathcal{M} \) it...
provides an element $\sigma(\pi) \in \mathcal{H}(S)$, living in the unramified cohomology of $S$. In the classical case when $\mathcal{M} = BG$ they were studied by many authors in relation to rationality problems and essential dimension, see for example [GMS03, Gar09, Gui08, Mer16].

The natural next steps after my computations in [Pir15, Pir17, Pir16] would be to compute the cohomological invariants of $\mathcal{H}_g$ for all odd $g$ and of the stack $\mathcal{M}_3$ of smooth genus three curves. I plan to attack these questions using new presentations of these stacks that are being developed by Andrea Di Lorenzo, a student of Vistoli, as part of his PhD thesis.

**Project 1** (joint with A. Di Lorenzo). Compute the cohomological invariants of $\mathcal{H}_g$ for all odd $g$ and of $\mathcal{M}_3$.

Studying the invariants of $\mathcal{M}_g$ for general $g$ will require a different approach. Let $T_g$ be the profinite completion of the $g$-th Teichmüller group. There is a map $\mathcal{M}_g \to BT_g$, which is an isomorphism from the point of view of étale homotopy type. This in particular induces maps $\mathcal{M}_g \to BG$ for all finite quotients $G$ of $T_g$. A natural subring of $\text{Inv}^*(\mathcal{M}_g)$ to study is the ring generated by the restrictions of the cohomological invariants of all such groups to those of $\mathcal{M}_g$.

**Project 2.** Study the subring of $\text{Inv}^*(\mathcal{M}_g)$ generated by the cohomological invariants of finite quotients of the Teichmüller group.

**Motivic classes of classifying stacks:**

In the late 2000s Ekedahl defined a Grothendieck ring of algebraic stacks $K_0(\text{Stk}/k)$, in analogy with the classical Grothendieck ring of algebraic varieties. Motivic invariants factor through it, making it an important object of study.

The “expected class formula” for the class of $BG$ predicts that it should be $\{G\}^{-1}$ when $G$ is connected and 1 when $G$ is finite. There are counterexamples for finite groups, and it is conjectured that the formula should not hold in general for connected groups either. This problem seems to be morally related to a major problem in group theory, Noether’s problem for connected algebraic groups.

The class of $BG$ has been computed for $\text{PGL}_2$, $\text{PGL}_3$ and $\text{SO}_n$ [Be16, DY16, TV17]. In a joint paper with Mattia Talpo [PT17] we showed that the problem of whether $B\text{Spin}_n$ satisfies the expected class formula boils down to the same problem for a certain finite subgroup $\Delta_n \subset \text{Spin}_n$. We conjecture that $B\text{Spin}_n$ should violate the formula for $n \geq 15$.

**Project 3** (joint with M. Talpo). Prove that $B\text{Spin}_n$ fails to satisfy the expected class formula for some $n \geq 15$.

**Essential dimension of representation functors:**

Consider a functor $F : (\text{Fields}/k) \to (\text{Sets})$. The essential dimension of $x \in F(K)$ is the minimal number of independent parameters needed to define (an object pulling back to) $x$.

The notion has proven to be an interesting measure of complexity, and it has been studied both for the functors of principal $G$-bundles (for example in [BRV10, CM14] for $G = \text{Spin}_n$) and for the functors of points of isomorphism classes of smooth curves and abelian varieties [BRV11].

Recently Benson, Karpenko, Reichstein and Pevtsova [KR15, BR17] studied the essential dimension of representations of finite groups and algebras, and Biswas, Dhillon and Hoffmann studied the essential dimension of vector bundles over a curve [BDH15]. Their methods, and new ones, can be applied to study the case of finitely generated algebras and quivers, which is the aim of a joint project with Z. Reichstein, F. Scavia and A. Vistoli.

**Project 4** (joint with Z. Reichstein, F. Scavia, A. Vistoli). Study the essential dimension of representation functors of finitely generated algebras and finite quivers.
Grothendieck categories and birational geometry:

A famous theorem of Gabriel [Gab62] states that a Noetherian scheme $X$ can be retrieved from its category of coherent sheaves. Starting from a recent paper of Meinhardt and Partsch [MP14] me and J. Calabrese [CP17] prove a birational extension of the theorem. Consider a scheme $X$ of finite type over a field $k$. If we take the subcategory $\text{Coh}^{\geq c}(X) \subset \text{Coh}(X)$ of sheaves supported in codimension $c$ or more, the quotient $\text{Coh}(X)/\text{Coh}^{\geq c}(X)$ recovers the isomorphism class of $X$ up to subsets of codimension at least $c + 1$.

Our construction appears well suited to apply to non-commutative algebraic geometry as defined by Artin, Smith, Van den Bergh et al. Birational geometry of non-commutative schemes is an emerging research area (see for example [PV16]), where many basic questions remain open. For example, to our knowledge there are no nontrivial examples of birational phenomena in codimension greater than one, such as non-commutative flops.

Project 5 (joint with J. Calabrese). Define a non-commutative notion of isomorphism in codimension $k$ using our construction and prove that it has properties resembling those of commutative birational geometry.

Relative invariants of universal families

Given a moduli stack $\mathcal{M}$, there is a universal family $\mathcal{C} \to \mathcal{M}$ over it which is just as relevant. Given a family $C \to S$, an invariant for $\mathcal{M}$ (e.g., an element of the Picard group) will pull back to an invariant for $S$ depending on $C$, while an invariant for $\mathcal{C}$ will pull back to an invariant of $C$.

In a joint paper with R. Fringuelli [FP16] we proved that for the universal Abelian variety $\mathcal{X}_g$ over $\mathcal{A}_g$, the relative Picard group $\text{Pic}(\mathcal{X}_{g,n})/\text{Pic}(\mathcal{X}_{g,n})$ is a direct sum of $(\mathbb{Z}/n\mathbb{Z})^{2g}$ and a free module generated by a canonical line bundle whose corresponding divisor is the theta divisor when $n$ is even and two times the theta divisor when $n$ is odd.

There are recent computations [BH12, BLS98] of Picard groups and Brauer groups of the moduli stack principal bundles over a fixed curve. I have a joint project with Roberto Fringuelli to “globalize” some of these results to the moduli stacks of principal $G$-bundles $\text{Bun}_G$ over $\mathcal{M}_g$.

Project 6 (joint with R. Fringuelli). Compute the relative Picard group and Brauer group of $\text{Bun}_G$ over $\mathcal{M}_g$ for $G$ a semisimple linear algebraic group.

References


