Research statement

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Introduction

My main area of research is Algebraic Geometry. More precisely, my areas of interest are moduli spaces (and moduli stacks) with particular regard to their arithmetic and cohomological properties, algebraic groups and their representations, cycle theories, motivic classes, essential dimension and abelian categories. The fundamental connecting tissue between these areas is the use of *algebraic stacks*.

To study a geometric object X one often needs to consider families $\mathscr{X} \to S$ varying "continuously" over a base S, in which X is embedded as the inverse image of a special point $s_0 \in S$. This idea led, among many advances, to the powerful tool of moduli spaces. These are spaces (i.e. topological, differential, algebraic varieties...) whose points correspond to isomorphism classes of a given type of object. A moduli space M is *fine* if families over S of the objects it parametrizes correspond $1:1$ to maps from S to M. The very first examples of moduli spaces are the Grassmannian varieties $\mathrm{Gr}(V, r)$, parametrizing r-dimensional subspaces of a vector space V.

Moduli spaces have proven to be important not only in geometry (both algebraic, analytic and differential), but also in physics, where they have prominent roles in string theory (where the moduli spaces involved are actually algebro-geometric objects) and quantum field theory.

Since the late 1950s it became apparent that moduli spaces were insufficient for studying algebro-geometric objects with non-trivial automorphism, and in fact these are a fundamental obstruction to the existence of a fine moduli space. Moreover, if one constructs a moduli space anyway, it will be often singular even if the objects being parametrized are smooth and nicely behaved.

These problems were solved thanks to the introduction of algebraic stacks. One can think of an algebraic stack as an algebraic variety where the points have intrinsic stabilizer groups, corresponding to the automorphisms of the objects being parametrized. Algebraic stacks produce an algebro-geometric version of orbifolds, and also of classifying spaces BG in topology. Moduli *stacks* regain the "fine" property, as well as smoothness. A prominent example is the moduli stack $\mathscr{M}_g$ of smooth algebraic curves of genus g, constructed by Deligne and Mumford [DM69], which is smooth and comes with a universal family $\mathscr{C}_g \to \mathscr{M}_g$ such that given a family of curves $C \to S$ there is a unique map $f : S \to \mathscr{M}_g$ with $f^*\mathscr{C}_g = C$. For comparison, the (coarse) moduli space M_g is not smooth and does not admit a universal family $C_g \to M_g$. These examples and many others have made algebraic stacks a central object of study.

Past research

My work has concentrated on constructing and computing invariants for algebraic stacks. Functorial invariants for moduli stacks are especially useful, as they will automatically provide invariants for the families of objects being parametrized. Prime examples of invariants which are of interest are the Picard group of line bundles, the Chow groups (which are an algebraic version of singular homology), and étale cohomology with various kind of coefficients.

A majority of my research [Pir15, Pir17, Pir16] regards *Cohomological invariants*, a theory of arithmetic invariants which were classically associated with algebraic groups. The starting point of my investigation was to note that the cohomological invariants $\mathrm{Inv}^\bullet(G)$ of an algebraic group G should really be regarded as invariants of the corresponding algebraic stack BG, which like its topological counterpart classifies principal G-bundles. Starting from this observation it was natural to extend the theory to any algebraic stack. I then computed the cohomological invariants of the stacks of elliptic curves $\mathscr{M}_{1,1}$, of the stack of smooth genus two curves $\mathscr{M}_2$ and of the stacks $\mathscr{H}_g$ of smooth hyperelliptic curves of genus g for all even g and for $g = 3$.

I also studied the Picard groups of universal families of Abelian varieties [FP16] (joint with R.Fringuelli) and the motivic classes (that is, the classes in an appropriate Grothendieck ring of isomorphism classes of algebraic stacks $K_0(\mathrm{Stk}/k)$) of the classifying spaces BSpin_n of Spin_n-principal bundles [PT17] (joint with M.Talpo). Finally, I have a work in progress with J. Calabrese, relating certain quotients of the abelian categories of coherent sheaves on a scheme with its birational geometry, and a new project with Z.Reichstein and A.Vistoli regarding the essential dimension of representation functors.

Cohomological invariants

Consider an algebraic group G, and let Tors_G be the functor $\mathrm{Tors}_\mathrm{G} : (\mathrm{Fields}/k) \to (\mathrm{Sets})$ sending a field K/k to the set of isomorphism classes of principal G-bundles (more commonly called *torsors* in algebraic geometry) over K. In the modern definition [GMS03], coined by Serre and Rost, a *cohomological invariant* $\alpha \in \mathrm{Inv}^\bullet(\mathrm{G})$ is a natural transformation between Tors_G and the Galois cohomology (with coefficients in μ_n) functor $\mathrm{H}^\bullet : (\mathrm{Fields}/k) \to (\mathrm{Sets})$.

Cohomological invariants can be thought of as an arithmetic equivalent to characteristic classes. They were studied in relation to both rationality problems and essential dimension by Serre, Rost, Merkurjev, Garibaldi, Totaro and many others.

It is natural to view the cohomological invariants of G as invariants of the classifying stack BG, as by definition the functor of isomorphism classes of G-torsors is the functor of (isomorphism classes of) points of BG.

I extended the definition to any algebraic stack [Pir15, Def. 1.1] by defining a cohomological invariant of an algebraic stack $\mathscr{M}$ as a natural transformation from the functor of points $\mathrm{Pts}_{\mathscr{M}} : (\mathrm{Fields}/k) \to (\mathrm{Sets})$ to $\mathrm{H}^\bullet$ satisfying a natural continuity condition. This recovers the classical definition when $\mathscr{M} = \mathrm{BG}$.

When the stack $\mathscr{M}$ is smooth, I proved [Pir15, Thm. 4.4] that the ring of cohomological invariants is equal to the sheafification of étale cohomology in an appropriate Grothendieck topology. This makes the ring of cohomological invariants a natural extension of unramified cohomology to algebraic stacks. Moreover, when $\mathscr{M}$ is a smooth quotient stack I proved [Pir17, Prop. 2.10] that the ring of cohomological invariants is equal to the zero-dimensional part of the G-equivariant Chow ring with coefficients [Ros96, Gui08] in $\mathrm{H}^\bullet$, making it the main tool for computing cohomological invariants.

Computing the cohomological invariants of $\mathscr{M}_\mathbf{g}$

Using techniques coming from the study of equivariant Chow rings [EG96, MV06] and presentations produced by Arsie and Vistoli [AV04], and the theory of equivariant Chow rings with coefficients, I computed the cohomological invariants of the stack $\mathscr{M}_{1,1}$ of elliptic curves [Pir15, Thm. 5.1], the stacks $\mathscr{H}_g$ of hyperelliptic curves of genus g when g is even [Pir17, Thm. 4.1] and of the stack $\mathscr{H}_3$ of hyperelliptic curves of genus three [Pir16, Thm. 3.12]. The natural next steps would be to compute the cohomological invariants of $\mathscr{H}_g$ for all odd g and of the stack $\mathscr{M}_3$ of smooth genus three curves (note that $\mathscr{H}_2 = \mathscr{M}_2$). I plan to attack these questions using new presentations of these stacks that are being developed by Andrea Di Lorenzo, a student of Vistoli, as part of his Phd thesis.

Project 1 (joint with A. Di Lorenzo)**.** Compute the cohomological invariants of $\mathscr{H}_g$ for all odd g and of $\mathscr{M}_3$.

These techniques cannot be applied directly to $\mathscr{M}_g$ when g is high enough, as these stacks become of general type and this makes the existence of a "good" quotient stack presentation (i.e. one that makes for easy equivariant computations) for $\mathscr{M}_g$ unlikely. One way to get around this might be the following. Let T_g be the profinite completion of the g-th Teichmüller group. There is a map $\mathscr{M}_g \to \mathrm{BT}_g$, which is an isomorphism from the point of view of étale homotopy type. This in particular induces maps $\mathscr{M}_g \to \mathrm{BG}$ for all finite quotients G of T_g. A natural subring of $\mathrm{Inv}^\bullet(\mathscr{M}_g)$ to study is the ring generated by the restrictions of the cohomological invariants of all such groups to those of $\mathscr{M}_g$.

Project 2. Study the subring of $\mathrm{Inv}^\bullet(\mathscr{M}_g)$ generated by the cohomological invariants of finite quotients of the Teichmüller group.

Cycle theories for algebraic stacks

The main tool up to now for computing cohomological invariants is the theory of equivariant Chow groups with coefficients. This has the shortcoming of being only defined for quotient stacks. For ordinary Chow groups, the equivariant theory was extended to arbitrary algebraic stacks by Kresch [Kre99], but his approach seems unlikely to extend to Chow groups with coefficients.

An alternative approach would be, given a stack $\mathscr{M}$, to consider a smooth-Nisnevich covering ([Pir15, def 3.2]) $X \to \mathscr{M}$, where X is a scheme, and construct a double complex $C^i(X^j)$ where $X^j := X \times_{\mathscr{M}} \ldots \times_{\mathscr{M}} X$ is the j-fold fiber product of X over $\mathscr{M}$. Then we could define the cohomology of this double complex to be the Chow groups with coefficients for $\mathscr{M}$. A similars approach has been already used for higher Chow groups by Joshua in [Jo02] and [Jo02b], but the result is not completely satisfactory (the groups obtained have been shown to be independent of the covering only when the stack is Deligne-Mumford) due to the lack of functoriality of cycles. One way to get around this (at least for smooth stacks) which I'm working on should be to create a theory of operational Chow groups with coefficients, which are by definition functorial, and use it to prove independence of the covering by exploiting the isomorphism between operational and ordinary theories for smooth stacks.

This might also provide an approach to extend different cycle theories, such as higher Chow groups and Fasel's Chow-Witt groups, to general smooth algebraic stacks.

Project 3. Devise a non-equivariant approach to Chow groups with coefficients for smooth algebraic stacks.

Relative invariants of universal families

Given a moduli stack $\mathscr{M}$, there is a universal family $\mathscr{C} \to \mathscr{M}$ over it which is just as relevant. Given a family $C \to S$, an invariant for $\mathscr{M}$ (e.g. an element of the Picard group) will pull back to an invariant for S depending on C, while an invariant for $\mathscr{C}$ will pull back to an invariant of C.

An important type of question is, given a theory of invariants F, to compute the relative group $\mathrm{F}(\mathscr{C})/\mathrm{F}(\mathscr{M})$ of invariants for $\mathscr{C}$ that do not come from $\mathscr{M}$. A major example of this is Franchetta's conjecture (proven by E.Arabarello and M.Cornalba [AC87]), which claims that the relative Picard group $\mathrm{Pic}(\mathscr{C}_g)/\mathrm{Pic}(\mathscr{M})$ is freely generated by the cotangent bundle $\omega_{\mathscr{C}_g}$.

In a joint paper with R.Fringuelli we proved [FP16, Thm. A] an analogous assertion for the universal Abelian variety $\mathscr{X}_g$ over $\mathscr{A}_g$, showing that $\mathrm{Pic}(\mathscr{X}_{g,n})/\mathrm{Pic}(\mathscr{A}_{g,n})$ is a direct sum of $(\mathbb{Z}/n\mathbb{Z})^{2g}$ and a free module generated by a canonical line bundle whose corresponding divisor is the theta divisor when n is even and two times the theta divisor when n is odd.

Recently there have been some results [BH12, BLS98] regarding the Picard groups and Brauer groups of principal bundles over a fixed curve. I have a joint project with Roberto Fringuelli to "globalize" some of these results to the moduli stacks of principal G-bundles BunG_g over $\mathscr{M}_g$, by adapting and expanding the ideas we used in our paper.

Project 4 (joint with R.Fringuelli)**.** Compute the relative Picard group and Brauer group of BunG_g over $\mathscr{M}_g$ for G a semisimple linear algebraic group.

Motivic classes of classifying stacks

In the late 2000s Ekedahl defined a Grothendieck ring of algebraic stacks $K_0(\mathrm{Stk}/k)$. Its elements are isomorphism classes of algebraic stacks, subject to three relations: a) if $\mathscr{U} \subseteq \mathscr{X}$ is an open immersion with closed complement $\mathscr{V}$ then $\{\mathscr{U}\} + \{\mathscr{V}\} = \{\mathscr{X}\}$ b) we have $\{\mathscr{X} \times_k \mathscr{Y}\} = \{\mathscr{X}\} \cdot \{\mathscr{Y}\}$ c) if $\mathscr{E} \to \mathscr{X}$ is a vector bundle of rank d then $\{\mathscr{E}\} = \{\mathbb{A}^d \times_k \mathscr{X}\}$.

Computing the class of the classifying stacks BG is an open problem. There is an "expected class formula" saying that the class of BG should be $\{\mathrm{G}\}^{-1}$ when G is connected 1 when G is finite. It holds when G is special, but there are counterexamples for finite groups, and the formula is expected to fail for connected groups too, even though no counterexample is known. This problem seems to be (at least morally) related to a major problem in group theory, Noether's problem for connected algebraic groups, which asks whether given a connected algebraic group G with a generically free representation V the quotient V/G is rational. A negative answer is expected for this question, but no example is known.

The class of BG has been computed for connected groups in the cases of PGL_2 and PGL_3 by Bergh [Be16], in the case of SO_n for odd n by Dhillon and Young [DY16] and in the case of SO_n for all n by Talpo and Vistoli [TV17]. In a joint paper with Mattia Talpo we computed [PT17, Thm. 3.1, 3.8] the classes of $\mathrm{BG}_2, \mathrm{BSpin}_7, \mathrm{BSPin}_8$, and showed [PT17, Thm. 4.5] that for any n the problem of whether BSpin_n satisfies the expected class formula boils down to the same problem for a certain finite subgroup $\Delta_n \subset \mathrm{Spin}_n$. We conjecture that BSpin_n should violate the formula for $n \geq 15$.

Project 5 (joint with M.Talpo)**.** Prove that BSpin_n fails to satisfy the expected class formula for some $n \geq 15$.

Grothendieck categories and birational geometry

A famous theorem of Gabriel [Ga62] states that a Noehterian scheme X can be retrieved from its category of coherent sheaves. This theorem has been extended in many directions, allowing X to be non-Noetherian or an algebraic space.

A recent paper of Meinhardt and Partsch [MP14] proves a new extension as a side result. Consider an irreducible projective scheme X. If we take the subcategory $\mathrm{Coh}^{\geq c}(X) \subset \mathrm{Coh}(X)$ of sheaves supported in codimension c or more, and take the quotient $\mathcal{C}^c(X) = \mathrm{Coh}(X)/\mathrm{Coh}^{\geq c}(X)$, we get a category consisting of coherent sheaves up to those supported in codimension c or higher. Meinhardt and Partsch show that when $c = 0, 1$ this category recovers X up to codimension strictly higher than c, that is if $\mathcal{C}^c(X) = \mathcal{C}^c(Y)$ then X and Y are isomorphic outside of subsets of codimension at least $c+1$.

I have a work-in-progress with John Calabrese where we prove that this is true for arbitrary schemes of finite type over a field k (not necessarily projective). That is, the category $\mathcal{C}^c(X)$ recovers the geometry of X up to subsets of codimension at least $c+1$. Our construction, being fundamentally categorical and based on the notion of minimal objects, might be well suited to apply to non-commutative algebraic geometry, as its point of view is basically to replace a scheme X with its category of quasi-coherent sheaves. Birational geometry of non-commutative schemes is an emerging research area, where many basic questions remain open. For example, to our knowledge there are no nontrivial examples of birational phenomena in codimension greater than one, such as non-commutative flops.

Project 6 (joint with J.Calabrese)**.** Define a non-commutative notion of isomorphism in codimension k using our construction and prove that it has properties resembling those of commutative birational geometry.

Essential dimension of representation functors

Consider a functor $F : (\text{Fields}/k) \to (\text{Sets})$. The essential dimension of $x \in F(K)$ is the minimal number of independent parameters needed to define (an object that pulls back to) x. More precisely, $\text{ed}(x)$ is equal to the minimal transcendence degree over k of a field $k' \subset K$ such that there is an element $x' \in F(k')$ which restricts to $x \in F(K)$.

The notion has proven to be an interesting measure of complexity, and it has been studied both for the functors of principal G-bundles (for example in [BRV10, CM14] for $G = \text{Spin}_n$) and for the functors of points of isomorphism classes of smooth curves and abelian varieties [BRV11].

Recently Benson, Karpenko, Reichstein and Pevtsova [KR15, BR17] studied the essential dimension of representations of finite groups and algebras, and Biswas, Dhillon and Hoffmann studied the essential dimension of vector bundles over a curve [BDH15]. Their methods, and new ones, can be applied to study the case of finitely generatedy algebras and quivers, which is the aim of a joint project with Z.Reichstein, F.Scavia and A.Vistoli.

A toy example is the case of $\text{A} = \mathbb{F}_r$, the free algebra in r generators, whose representations can also be thought as representations of the quiver with 1 vertex and n loops. A representation of $\mathbb{F}_r$ of dimension n is given (not uniquely) by a point $p \in \text{M}^r_{n,n}$. The adjoint action of PGL_n preserves the isomorphism class of a representation, and the points of the quotient space can be viewed as all characters of representations of $\mathbb{F}_r$. Using this, a computation by Reichstein and Vistoli proves that the essential dimension of the functor of n-dimensional representations of $\mathbb{F}_r$ is equal to $n^2(r-1)+1+\text{cd}(\text{PGL}_n)$, where $\text{cd}(\text{PGL}_n)$ is the canonical dimension (the minimum dimension of the image of a rational map to itself) of a versal PGL_n torsor. One can also obtain the same result using ideas from [BDH15]. The combination of the two methods should provide a general line of attack to the problem.

Project 7 (joint with Z.Reichstein, F.Scavia, A.Vistoli)**.** Study the essential dimension of representation functors of finitely generated algebras and quivers.

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