1) **Triple integrals**

From Exam W 2006 T1

\[
\int_0^1 \int_0^{1-x/2} \int_0^{4-2x-4z} f(x,y,z) \, dz \, dx \, dy
\]

1) Sketch the domain

2) Rewrite as \( \int \int \int f \, dz \, dx \, dy \)

We have:

\( 0 \leq x \leq 1 \)

\( 0 \leq z \leq 1 - x/2 \) plane parallel to y-axis

\( 0 \leq y \leq 4 - 2x - 4z \) plane

1) Sketch \( y = 4 - 2x - 4z \)

\( y + 2x + 4z = 4 \)

Find intercepts:

\( y = 4 \)

\( 2x = 4 \), \( x = 2 \)

\( 4z = 4 \), \( z = 1 \)

3) Note: the line in the \( xy \)-plane through \((2,0,0)\) and \((0,0,1)\) is common.

\( \text{the red line} \).
Guess 1:
Our solid is this tetrahedron in the first octant.

Correction: \( 0 \leq x \leq 1 \! \) 
\( x=1 \) is a plane parallel to y and z-axes.

Intersection of the plane \( x=1 \) with 
\[ y + 2x + 4z = 4. \]

Guess 2:

Compare to the original integral:
\[
\int_0^1 \int_0^{1-x/2} \int_0^{4-2x-4z} dz \, dy \, dx
\]

These limits describe the blue domain in the \( xz \)-plane.

2) Rewrite as 
\[
\int_D \int_0^{4-2x-4z} dz \, dy \, dx
\]

What is the domain in the \( xy \)-plane we need to integrate over? (Projection onto the \( xy \)-plane).

Here it coincides with the cross-section with the \( xy \)-plane.

The order of integration is \( dx \, dy \).
So, on the outside, get:

\[ \iiint_D \, dx \, dy \, dz \]

\[ \left\{ \begin{array}{c}
\int_0^1 \int_0^{4-2x} \int_0^{\frac{4-y}{2}} f(x,y,z) \, dz \, dy \\
+ \int_0^1 \int_0^{\frac{4-y}{2}} \int_0^{4-2x-y} f(x,y,z) \, dz \, dy
\end{array} \right. \]

Now, what about the inner integral?

\[ \text{given: } 0 \leq y \leq 4 - 2x - 4z \]

have to rewrite this constraint

\[ y = 4 - 2x - 4z \text{ so that } \]

\[ z = 4 - 2x - y. \]

Put it together:

\[ \int_0^1 \int_0^{4-2x-y} \int_0^{\frac{4-y}{2}} f(x,y,z) \, dz \, dy \\
+ \int_0^1 \int_0^{\frac{4-y}{2}} \int_0^{4-2x-y} f(x,y,z) \, dz \, dy. \]

Advice: If you are just given a solid

\[ f \text{ and asked to set up an integral, choose convenient order.} \]

1) If given inequalities, put the variable

that is most used on the outside.

\[ \text{e.g. } y \leq f(x,z) \rightarrow \text{put } x \text{ on the outside.} \]

2) Draw the domain

for the outside double integral. Choose convenient order.
Wording: “bounded below by” = “above”
“bounded above by” = “below”.

Solid bounded below by the cone and bounded above by the sphere.

Below the cone above the xy-plane bounded above by the cone, bounded below by the xy-plane.

Set up both of these

Suppose the sphere has radius $R$. The cone has angle $\alpha$ at the vertex.

1) inside the sphere, above the cone:

$$\iiint_{V} f(r, \theta, \phi) r^2 \sin \phi \, dr \, d\phi \, d\theta$$

from $dV$ in spherical coords.

2) below the cone, inside the sphere:

$$\iiint_{V} f(\rho, \phi, \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$
3) \[ \int \int \int f \cdot p^2 \sin \phi \, dp \, d\phi \, d\theta \]

(ky-plane: \( \psi = \frac{\pi}{2} \) in spherical).

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**Question:** When converting to spherical coordinates, traditionally, \( \phi \) is “special”:

\[
\begin{align*}
\phi &= \rho \cos \psi \\
x &= \rho \sin \psi \cos \phi \\
y &= \rho \sin \psi \sin \phi \\
z &= \rho \sin \phi \\
\end{align*}
\]

What if we have a cone around the y-axis?

Could “rename \( x, y, z \)”:\[
\begin{align*}
\bar{\phi}, \bar{\psi} \\
y &= \rho \cos \bar{\phi} \\
z &= \rho \sin \bar{\phi} \sin \bar{\psi} \\
x &= \rho \sin \bar{\phi} \cos \bar{\psi} \\
\end{align*}
\]

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**About cones and angles:**

\[
\begin{align*}
\frac{r^2 + x^2}{r^2} &= \frac{1}{\cos^2(\alpha)} \\
\frac{\sqrt{x^2+y^2}}{z} &= \tan(\alpha) \\
z &= \sqrt{c(k^2+y^2)} \\
\frac{\sqrt{x^2+y^2}}{z} &= \frac{1}{\sqrt{c}} \\
\end{align*}
\]

So, \( \frac{1}{\sqrt{c}} = \tan(\alpha) \)
Also,

\[(3-a)^2 + x^2 + y^2 = a^2\]

\[\sqrt{x^2 + x^2 + y^2} = \sqrt{2a}\]

\[\sqrt{f^2} = \sqrt{2a} \cos \theta\]

\[f = 2a \cos \theta\]

Important: Have the integral of \(ds\) inside the integral of \(d\theta\).

Question:

1) Find \(\text{the horizontal plane } z = c \text{ where the spheres intersect.}\)

2) Use cylindrical coordinates:

\[\text{spheres of radius } R \text{ intersect at "height" } c\]

i) Find \(r = \text{radius of the circle of intersection.}\)
\[ r = \sqrt{R^2 - c^2} \]

Volume of the sphere is given by:

\[ \iiint_D 1 \, dz \, dx \, dy \]

Projection of the cap onto the xy-plane:

\[ D \]

\[ = \iiint_{\text{cylindrical}} 1 \, r \, dr \, dz \, d\theta \]

\[ = \int_0^{2\pi} \int_0^{\sqrt{R^2 - c^2}} \int_0^{\sqrt{R^2 - r^2}} r \, dz \, dr \, d\theta \]

From Math 101:
1) Substitution (e.g., \( x^2 = u \), \( \frac{2x \, dx}{\sqrt{a^2 - x^2}} = du \))
2) Useful integrals:
   \[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx \]
   \[ \int \cos \theta \, d\theta, \quad \int \sin \theta \, d\theta, \quad \int \cos^2 \theta \, d\theta, \quad \int \sin^2 \theta \, d\theta \]
   Higher powers will give hints.
3) Integration by parts.
Given: \( z = f(x, y) \)  
\( x = 2 + t^2 \)  
\( y = t^3 \)  
\( f_x (2,1) = 5 \)  
\( f_y (2,1) = -2 \)  
\( f_{xx} (2,1) = 2 \)  
\( f_{xy} (2,1) = 1 \)

Find: \( \frac{d^2 z}{dt^2} \) at \( t = 1 \)  
(Note: \( z \) ends up being a function of single variable \( t \)).

\[ z''(t) \text{ at } t = 1 = z''(1) \]

\[ \frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \]

Note: When \( t = 1 \), get \( x = 2.1^2 = 2 \) \( y = 1^3 = 1 \)  
\( (x,y) = (2,1) \).

\[ \frac{dx}{dt} = 4t \]  
\[ \frac{dy}{dt} = 3t^2 \]

Plug this in:

\[ \frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot 4t + \frac{\partial f}{\partial y} \cdot 3t^2 \]  
(Note: too early to plug in values for \( f_x, f_y \)).

Differentiate

\[ \frac{d^2 z}{dt^2} = \frac{d}{dt} \left( \frac{\partial f}{\partial x} \right) \cdot 4t + \frac{\partial f}{\partial x} \cdot 4 \]

\[ + \frac{d}{dt} \left( \frac{\partial f}{\partial y} \right) \cdot 3t^2 + \frac{\partial f}{\partial y} \cdot 6t \]

\( = \left( \frac{\partial^2 f}{\partial x^2} \cdot \frac{dx}{dt} + \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{dy}{dt} \right) \cdot 4t \]

\( + \left( \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{dy}{dt} \right) \cdot 3t^2 \]

\( + \frac{\partial f}{\partial y} \cdot 6t \)
\[
= (f_{xx} \cdot 4 + f_{xy} \cdot 3t^2) \cdot 4t + 4f_x
\]
\[
+ (f_{xy} \cdot 4t + f_{yy} \cdot 3t^2) \cdot 3t^2 + 6t^2 f_y
\]

Now: plug in \( t = 1 \) into all \( f(x, y) \) functions of \((x, y)\).

Get: 
\[
(\ f_{xx} (2, 1) \cdot 4 + f_{xy} (2, 1) \cdot 3 \cdot 4
\]
\[
+ 4 f_x (2, 1) + (f_{xy} (2, 1) \cdot 4 + f_{yy} (2, 1) \cdot 3) \cdot 3
\]
\[
+ 6 f_y (2, 1)
\]

\[
= (2 \cdot 4 + 1 \cdot 3) \cdot 4 + 4 \cdot 5 + \ldots \ldots
\]

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**Lagrange** primitive question:

Find the distance from a point to a plane in space.

**EX:** Given \( 2z + x - y = 4 \)

Find the distance from \((3, 0, 2)\) to this plane.

Use analysis, not geometry:

\[
f(x, y, z) = (\text{dist from } (x, y, z) \text{ to } P)^2
\]
\[
= (x - 3)^2 + (y - 0)^2 + (z - 2)^2
\]

\[
= (x - 3)^2 + y^2 + (z - 2)^2
\]

\[ P \] is shortest for all distances from \( P \) to points on the plane.
Need to find min \( f \) subject to the constraint
\[ 2x + x - y = 4, \quad \text{and} \quad g(x, y, z) = 2x + x - y - 4 \]

\[ \nabla f = \lambda \nabla g \]

\[ \nabla f = \langle 2(x-3), \quad 2y, \quad 2(z-2) \rangle \]

\[ \nabla g = \langle 1, \quad -1, \quad 2 \rangle \]

Get:

\[ \begin{cases} 
2(x-3) = \lambda \\
2y = -\lambda \\
2(z-2) = 2\lambda \\
2x + x - y = 4
\end{cases} \]

Express everything through \( \lambda \), plug into the last eqn.

\[ \begin{cases} 
x = \frac{\lambda + 6}{2} \\
y = -\frac{\lambda}{2} \\
z = \frac{2\lambda + 4}{2} = \lambda + 2 \\
2(\lambda + 2) + \frac{\lambda + 6}{2} + \lambda = 4 \\
3\lambda + 7 = 4
\end{cases} \]

\[ \lambda = -\frac{3}{3} = -1. \]

\[ x = \frac{5}{2}, \quad y = \frac{1}{2}, \quad z = 1. \]

Answer: \( \sqrt{(\frac{5}{2} - 3)^2 + (\frac{1}{2} - 0)^2 + (1 - 2)^2} = \ldots \)

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