Triple integrals

A solid $E$ (filled in 3-dim object).

$f(x,y,z)$ - function of 3 variables defined on $E$

(examples: $f(x,y,z)$ - density of the material
$T(x,y,z)$ - temperature, solid = all the air

Define $\iiint_E f(x,y,z) \, dV$

with respect to volume.

Definition: using Riemann sums:

Chop $E$ (approximately) into small boxes

$(x^*_i, y^*_i, z^*_i)$

pick a point in each box

$\sum_{i,j,k} f(x^*_i, y^*_i, z^*_i) \frac{\Delta x \Delta y \Delta z}{\text{volume of the box}}$

Riemann sum for our integral.

(when $f$ is continuous)

Riemann sums have a limit as the boxes get smaller

the limit is called $\iiint_E f(x,y,z) \, dV$. 


To compute: set up as an iterated integral

(1) The shape of $E$ is encoded in the limits of integration.

Example: Find the average of $f(x,y,z) = x+y-2$ over the tetrahedron bounded by the coordinate planes and the plane $3x+2y+z = 6$

Note: average of a function:
- over a 3rd solid: $\frac{1}{\text{Vol}(E)} \iiint_E f(x,y,z) \, dV$
- over a 2nd region: $\frac{1}{\text{area}(D)} \iint_D f(x,y) \, dA$
- over an interval $[a,b]$: $\frac{1}{b-a} \int_a^b f(x) \, dx$
1. Volume of $E$:

2 ways:
1) think of $E$ as the solid under the graph of $z = h(x,y)$
   
   or 2) $\iiint_{E} 1 \, dV$.

   **Drawing planes:**
   
   $3x + 2y + z = 6$
   
   - look for intercepts with axes
   
   - think of whether the plane is parallel to something.

   $3x = 6$ (x-axis)

   $2y = 6$

   $z = 6$

   Our plane:
   
   graph of $z = 6 - 2y - 3x$.

   **Volume:** our solid: under the graph of $z = 6 - 2y - 3x$

   over the triangle $\triangle_{1,0,0}$
\[ V = \iiint_T (6 - 3x - 2y) \, dA \]
\[ = \int_0^2 \int_0^{3 - \frac{3}{2} x} (6 - 3x - 2y) \, dy \, dx \]

\[ (\text{should be the red triangle } T) \]

If we were computing the volume by a triple integral, plane \( z = 2 - \frac{3}{2} x \):

\[ V = \iiint_0^{3 - \frac{3}{2} x} (6 - 3x - 2y) \, dz \, dy \, dx \]

\[ = \int_0^2 \int_0^{3 - \frac{3}{2} x} (6 - 3x - 2y) \, dy \, dx \]

**evaluate from inside out**
We have our integral:

$$\int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} 1 \, dz \, dy \, dx$$

inside an integral with respect to x, so can use x, y for limits.

Outside limits have to be numbers.

Can use x for limits inside the integral with respect to x.

For the average:

$$\text{Average } (f) = \frac{1}{V} \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} f(x,y,z) \, dz \, dy \, dx$$

see above from the problem.

\[
= \frac{1}{V} \int_0^2 \int_0^{3-\frac{3}{2}x} x^2 + y^2 - \frac{1}{2} z^2 \left|_{z=0}^{6-3x-2y} \right. \, dy \, dx
\]

\[
= \frac{1}{V} \int_0^2 \int_0^{3-\frac{3}{2}x} \left( x(6-3x-2y) + y(6-3x-2y) - \frac{1}{2} (6-3x-2y)^2 \right) \, dy \, dx
\]
1. Set up, in any order, the integral representing the x-coordinate of the centroid of the solid bounded by the planes \( x = 0, \ y = 0, \ z = 0, \ y + x = 1, \) and the parabolic cylinder \( z = 1 - x^2. \)

\[ z = 1 - x^2 \text{ consists of lines parallel to the } y\text{-axis} \]

\( y \) is not in the equation

\[ M = \int_0^1 \int_0^{1-x^2} \int_0^{4-x} \int_0^y dz \, dy \, dx \]

For \( x, \) see p. 3

\[ \text{Change the order of integration so that the iterated integral is of the form } \iiint - \, dx \, dy \, dz \]

Will do problem 2 on Monday.

See next page for a complete picture of the solid; as a hint — think about it before Monday.

Note also that the order of integration is \( #1 \) on the next page is different from the order on this page.
1. Set up, in any order, the integral representing the x-coordinate of the centroid of the solid bounded by the planes \( x=0, y=0, z=0, y+x=1, \) and the parabolic cylinder \( z=1-x^2. \)

\[
M \ = \ \iint_0^1 \left( \int_0^{1-x^2} \int_0^{1-x} 1 \, dy \, dz \right) \, dx
\]

Solid: under the cylinder \( z=1-x^2 \)
so \( 0 \leq z \leq 1-x^2 \)
\( 0 \leq y \leq 1-x \)

2. Change the order of integration so that the iterated integral is of the form \( \iiint \) form \( \int \, dx \, dy \, dz \)

Note for #1: we notice that limits for \( y, z \) depend only on \( x \). So if we fix \( x \), limits for \( y, z \) are constant. Then cross-sections of our solid parallel to the \( yz \)-plane are rectangles!
\[
\bar{x} = \frac{1}{M} \iiint_{0}^{1-x} x \cdot 1 \, dz \, dy \, dx, \quad \bar{y} = \frac{1}{M} \iiint_{0}^{1-x} y \cdot 1 \, dz \, dy \, dx
\]
\[
\bar{z} = \frac{1}{M} \iiint_{0}^{1-x} 1 \cdot z \, dz \, dy \, dx
\]

*Note*: In this example, could say that

\[M = \text{Volume under the graph of } z = 1-x^2 \text{ over } \]

and set it up as

\[
\int_{0}^{1-x} \int_{0}^{1-x^2} (1-x^2) \, dy \, dx
\]

but say, \(z\)-coordinate of the center of mass is

\[
\frac{1}{M} \iiint_{0}^{1-x} z \, dz \, dy \, dx \text{ cannot be obtained from such double integral.}
\]